Nullstellensatz theorem for skew PBW extensions

Oswaldo Lezama
jolezamas@unal.edu.co
Seminario de Álgebra Constructiva - SAC^2
Departamento de Matemáticas
Universidad Nacional de Colombia, Sede Bogotá

Algebraic groups, their friends and relations
St. Petersburg Department of Steklov Mathematical Institute
of Russian Academy of Sciences, September 18-23, 2022
Dedicated to Nikolai Vavilov, my friend and Ph.D. thesis co-supervisor

Abstract

In this talk we define the algebraic sets and the ideal of points for bijective skew PBW extensions with coefficients in left Noetherian domains. Some properties of affine algebraic sets of commutative algebraic geometry will be extended, in particular, a Zariski topology will be constructed. Assuming additionally that the extension is quasi-commutative with polynomial center and the ring of coefficients is an algebraically closed field, we will prove an adapted version of Hilbert's Nullstellensatz theorem that covers the classical one. The Gröbner bases of skew PBW extensions will be used for defining the algebraic sets and for proving the main theorem. Many key algebras and rings coming from mathematical physics and non-commutative algebraic geometry are skew PBW extensions (see **Fajardo, W., Gallego, C., Lezama, O., Reyes, A., Suárez, H., and Venegas, H.**, *Skew PBW Extensions: Ring and module theoretic properties, matrix and Gröbner methods, applications*, Algebra and Applications 28, Springer, 2020).

Contents

Key words and phrases. Non-commutative algebraic geometry, algebraic sets, ideals of points, Hilbert's Nullstellensatz theorem, non-commutative Gröbner bases, skew PBW extensions.

2020 *Mathematics Subject Classification.* Primary: 16S38. Secondary: 16N40, 16S36, 16Z05.

1 Introduction

- The **ring-theoretic** version of the Hilbert's Nullstellensatz theorem for skew PBW extensions has been considered in **Reyes, A. and Hernández-Mogollón, R.**, *A survey on some characterizations of Hilbert's Nullstellensatz for some non-commutative rings of polynomial type*, Ing. Cienc., 16, 2020, 27-52 ([14]), following the approach given in **McConnell, J. and Robson, J.,** *The Nullstellensatz and generic flatness*, Perspectives in Ring Theory, 1988, pp. 227-232 ([12]). **The algebraic characterization of the theorem presented by the authors of [14] does not use the notion of variety** (Theorem 3.1, [14]).
- Applying the algebraic sets and the ideal of points introduced in Definition 3.3 and Theorem 3.4, we present in this talk the classical version of this important theorem for quasi-commutative bijective skew PBW extensions of algebraically closed fields. Our version covers the classical Nullstellensatz theorem of commutative algebraic geometry.

For completeness, we recall some notions related to prime ideals of an arbitrary ring. By a ring we mean associative ring with unit.

Definition 1.1. *Let S be a ring and I, P be two-sided ideals of S, with $P \neq S$.*

(i) *P is a* ***prime ideal*** *of S if for any left ideals L, J of S the following condition holds: $LJ \subseteq P$ if and only if $L \subseteq P$ or $J \subseteq P$.*

(ii) *The* ***radical*** *of I, denoted $\sqrt{I}$, is the intersection of all prime ideals of S containing I.*

(iii) *An element $a \in S$ is* ***I-strongly nilpotent*** *if for any given sequence $\mathcal{S} := \{a_i\}_{i \geq 1}$ of elements of S, with $a_1 := a$ and $a_{i+1} \in a_i S a_i$, there exists $m(\mathcal{S}) \geq 1$ such that $a_{m(\mathcal{S})} \in I$. We say that a is* ***I-nilpotent*** *if there exists $m \geq 1$ such that $a^m \in I$.*

(iv) *P is* ***completely prime*** *if the following condition holds for any $a, b \in S$: $ab \in P$ if and only if $a \in P$ or $b \in P$.*

(v) *P is* ***completely semiprime*** *if the following condition holds for any $a \in S$: $a^2 \in P$ if and only if $a \in P$.*

- It is clear that if $a \in S$ is I-strongly nilpotent, then a is I-nilpotent. If $a \in Z(S)$, then the converse is true.
- If P is completely prime, then P is completely semiprime.
- By induction on m it is easy to show that P is completely semiprime if and only if the following condition holds: For any $a \in S$ and any $m \geq 1$, $a^m \in P$ if and only if $a \in P$.

Proposition 1.2. *Let S be a ring and I be a two-sided ideal of S. Then,*

$$\sqrt{I} = \{a \in S \mid a \text{ is } I\text{-strongly nilpotent}\}.$$

2 Skew PBW extensions

In this section we recall some basic facts about the class of noncommutative rings of polynomial type known as skew PBW extensions. In particular, we include the ingredients of the Gröbner theory of skew PBW extensions needed in the next section. For more details see

Fajardo, W., Gallego, C., Lezama, O., Reyes, A., Suárez, H., and Venegas, H., *Skew PBW Extensions: Ring and module theoretic properties, matrix and Gröbner methods, applications*, Algebra and Applications 28, Springer, 2020 ([10], Chapters 1, 2, 3, 13).

Definition 2.1 ([8],[10])**.** *Let R and A be rings. We say that A is a* ***skew** PBW **extension of*** R *(also called a $\sigma - PBW$ extension of R) if the following conditions hold:*

(i) $R \subseteq A$.

(ii) *There exist finitely many elements $x_1, \dots, x_n \in A$ such A is an R-free left module with basis*

$$\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}, \text{ with } \mathbb{N} := \{0, 1, 2, \dots\}.$$

In this case we say that A is a ***ring of left polynomial type*** *over R with respect to $\{x_1, \dots, x_n\}$. The set $\mathrm{Mon}(A)$ is called the set of* ***standard monomials*** *of A.*

(iii) *For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that*

$$x_i r - c_{i,r} x_i \in R. \tag{2.1}$$

(iv) *For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that*

$$x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n. \tag{2.2}$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \dots, x_n\rangle$.

Associated to a skew PBW extension $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ there are n injective endomorphisms $\sigma_1, \dots, \sigma_n$ of R and σ_i-derivations, as the following proposition shows.

Proposition 2.2 ([8], Proposition 3)**.** *Let A be a skew PBW extension of R. Then, for every $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that*

$$x_i r = \sigma_i(r)x_i + \delta_i(r),$$

for each $r \in R$.

Two remarkable particular cases of skew PBW extensions are recalled next.

Definition 2.3 ([10], Chapter 1)**.** *Let A be a skew PBW extension.*

(a) *A is **quasi-commutative** if conditions (iii) and (iv) in Definition 2.1 are replaced by*

(iii′) *For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists a $c_{i,r} \in R - \{0\}$ such that*

$$x_i r = c_{i,r} x_i. \tag{2.3}$$

(iv′) *For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that*

$$x_j x_i = c_{i,j} x_i x_j. \tag{2.4}$$

(b) *A is **bijective** if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i, j \leq n$.*

Many important algebras and rings coming from mathematical physics and non-commutative algebraic geometry are particular examples of skew PBW extensions:

- **Habitual ring of polynomials in several variables**.
- Weyl algebras, enveloping algebras of finite dimensional Lie algebras, algebra of q-differential operators, many important types of Ore algebras, in particular, single Ore extensions, algebras of diffusion type, additive and multiplicative analogues of the Weyl algebra, dispin algebra $\mathcal{U}(osp(1,2))$, quantum algebra $\mathcal{U}'(so(3,K))$, Woronowicz algebra $\mathcal{W}_\nu(\mathfrak{sl}(2,K))$, Manin algebra $\mathcal{O}_q(M_2(K))$, coordinate algebra of the quantum group $SL_q(2)$, q-Heisenberg algebra $\mathbf{H}_n(q)$, Hayashi algebra $W_q(J)$, differential operators on a quantum space $D_{\mathbf{q}}(S_{\mathbf{q}})$, Witten's deformation of $\mathcal{U}(\mathfrak{sl}(2,K))$, multiparameter Weyl algebra $A_n^{Q,\Gamma}(K)$, quantum symplectic space $\mathcal{O}_q(\mathfrak{sp}(K^{2n}))$, some quadratic algebras in 3 variables, some 3-dimensional skew polynomial algebras, particular types of Sklyanin algebras, homogenized enveloping algebra $\mathcal{A}(\mathcal{G})$, Sridharan enveloping algebra of 3-dimensional Lie algebra $\mathcal{G}$, among many others. For a precise definition of any of these rings and algebras see [9] and [10].
- The skew PBW extensions has been intensively studied in the last years (see [10]).

Next we fix some notation and a monomial order in A (see [10], Chapter 1).

Definition 2.4. *Let A be a skew PBW extension of R with endomorphisms σ_i as in Proposition 2.2, $1 \leq i \leq n$.*

(i) *For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\boldsymbol{\sigma^{\alpha}} := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, then $\boldsymbol{\alpha + \beta} := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.*

(ii) *For $X = x^{\alpha} \in \mathrm{Mon}(A)$, $\mathbf{exp}(\boldsymbol{X}) := \alpha$ and $\mathbf{deg}(\boldsymbol{X}) := |\alpha|$.*

(iii) *Let $0 \neq f \in A$. If $t(f)$ is the finite set of terms that conform f, i.e., if $f = c_1X_1 + \cdots + c_tX_t$, with $X_i \in \mathrm{Mon}(A)$ and $c_i \in R - \{0\}$, then $\boldsymbol{t(f)} := \{c_1X_1, \ldots, c_tX_t\}$.*

(iv) *Let f be as in* (iii), *then $\mathbf{deg}(\boldsymbol{f}) := \max\{\deg(X_i)\}_{i=1}^{t}$.*

- In $\mathrm{Mon}(A)$ we define

$$x^\alpha \succeq x^\beta \Longleftrightarrow \begin{cases} x^\alpha = x^\beta \\ \text{or} \\ x^\alpha \neq x^\beta \text{ but } |\alpha| > |\beta| \\ \text{or} \\ x^\alpha \neq x^\beta, |\alpha| = |\beta| \text{ but } \exists\ i \text{ with } \quad \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i. \end{cases}$$

  It is clear that this is a total order on $\mathrm{Mon}(A)$, called ***deglex*** order. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$.

- Each element $f \in A - \{0\}$ can be represented in a unique way as $f = c_1 x^{\alpha_1} + \cdots + c_t x^{\alpha_t}$, with $c_i \in R - \{0\}$, $1 \leq i \leq t$, and $x^{\alpha_1} \succ \cdots \succ x^{\alpha_t}$.

- We say that x^{α_1} is the ***leading monomial*** of f and we write $lm(f) := x^{\alpha_1}$; c_1 is the ***leading coefficient*** of f, $lc(f) := c_1$, and $c_1 x^{\alpha_1}$ is the ***leading term*** of f denoted by $lt(f) := c_1 x^{\alpha_1}$. We say that f is ***monic*** if $lc(f) := 1$. If $f = 0$, we define $lm(0) := 0$, $lc(0) := 0$, $lt(0) := 0$, and we set $X \succ 0$ for any $X \in \mathrm{Mon}(A)$.

The next proposition complements Definition 2.1.

Proposition 2.5 ([8],[10])**.** *Let A be a ring of a left polynomial type over R w.r.t. $\{x_1, \dots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions hold:*

(a) *For every $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$ there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R - \{0\}$ and $p_{\alpha,r} \in A$ such that*

$$x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}, \tag{2.5}$$

where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. Moreover, if r is left invertible, then r_α is left invertible.

(b) *For every $x^\alpha, x^\beta \in \mathrm{Mon}(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that*

$$x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{2.6}$$

where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

We conclude this subsection recalling some of the **main ingredients of the Gröbner theory of skew PBW extensions**, namely, **the notion of reduction, the Division Algorithm and the notion of Gröbner basis of a left ideal of** A. For all details see [10], Chapter 13.

For the condition (ii) in Definition 2.8 below, some natural computational conditions on R will be assumed.

Definition 2.6. *A ring R is **left Gröbner soluble** (LGS) if the following conditions hold:*

(i) *R is left noetherian.*

(ii) *Given $a, r_1, \dots, r_m \in R$ there exists an algorithm which decides whether a is in the left ideal $Rr_1 + \cdots + Rr_m$, and if so, finds $b_1, \dots, b_m \in R$ such that $a = b_1r_1 + \cdots + b_mr_m$.*

(iii) *Given $r_1, \dots, r_m \in R$ there exists an algorithm which finds a finite set of generators of the left R-module of syzygies*

$$\operatorname{Syz}_R[r_1 \ \cdots \ r_m] := \{(b_1, \dots, b_m) \in R^m \mid b_1r_1 + \cdots + b_mr_m = 0\}.$$

Definition 2.7. *Let $x^\alpha, x^\beta \in \text{Mon}(A)$. We say that x^α* ***divides*** *x^β, denoted by $x^\alpha \mid x^\beta$, if there exists a unique $x^\theta \in \text{Mon}(A)$ such that $x^\beta = lm(x^\theta x^\alpha) = x^{\theta+\alpha}$ and hence $\beta = \theta + \alpha$.*

Definition 2.8. *Let F be a finite set of nonzero elements of A, and let $f, h \in A$. We say that f* ***reduces to*** *h* ***by*** *F* ***in one step****, denoted $f \xrightarrow{F} h$, if there exist elements $f_1, \dots, f_t \in F$ and $r_1, \dots, r_t \in R$ such that*

(i) *$lm(f_i) \mid lm(f)$, $1 \leq i \leq t$, i.e., there exists an $x^{\alpha_i} \in \text{Mon}(A)$ such that $lm(f) = lm(x^{\alpha_i} lm(f_i))$, i.e., $\alpha_i + \exp(lm(f_i)) = \exp(lm(f))$.*

(ii) *$lc(f) = r_1 \sigma^{\alpha_1}(lc(f_1)) c_{\alpha_1, f_1} + \cdots + r_t \sigma^{\alpha_t}(lc(f_t)) c_{\alpha_t, f_t}$, where c_{α_i, f_i} are defined as in Proposition 2.5, i.e., $c_{\alpha_i, f_i} := c_{\alpha_i, \exp(lm(f_i))}$.*

(iii) *$h = f - \sum_{i=1}^{t} r_i x^{\alpha_i} f_i$ (this equation means that the leading term of f has been eliminated).*

*We say that f **reduces** to h by F, denoted $f \xrightarrow{F}_{+} h$, if there exist $h_1, \ldots, h_{t-1} \in A$ such that*

$$f \xrightarrow{\quad F \quad} h_1 \xrightarrow{\quad F \quad} h_2 \xrightarrow{\quad F \quad} \cdots \xrightarrow{\quad F \quad} h_{t-1} \xrightarrow{\quad F \quad} h.$$

*f is **reduced** (also called **minimal**) w.r.t. F if $f = 0$ or there is no one step reduction of f by F, i.e., one of the conditions (i) or (ii) fails. Otherwise, we will say that f is **reducible** w.r.t. F. If $f \xrightarrow{F}_{+} h$ and h is reduced w.r.t. F, then we say that h is a **remainder** for f w.r.t. F.*

By definition we will assume that $0 \xrightarrow{F} 0$.

Proposition 2.9 (Division algorithm)**.** *Let* $F = \{f_1, \ldots, f_t\}$ *be a finite set of nonzero polynomials of* A *and* $f \in A$*, then there exist polynomials* $q_1, \ldots, q_t, h \in A$*, with* h *reduced w.r.t.* F*, such that* $f \xrightarrow{F}_{+} h$ *and*

$$f = q_1 f_1 + \cdots + q_t f_t + h,$$

with

$$lm(f) = \max\{lm(lm(q_1)lm(f_1)), \ldots, lm(lm(q_t)lm(f_t)), lm(h)\}.$$

Definition 2.10. *Let $I \neq 0$ be a left ideal of A and let G be a nonempty finite subset of nonzero polynomials of I. G is a **Gröbner basis** for I if each element $0 \neq f \in I$ is reducible w.r.t. G.*

Proposition 2.11. *Let $I \neq 0$ be a left ideal of A. Then,*

(i) *If G is a Gröbner basis for I, then $I = \langle G\}$ (the left ideal of A generated by G).*

(ii) *Let G be a Gröbner basis for I. If $f \in I$ and $f \xrightarrow{G}_{+} h$, with h reduced, then $h = 0$.*

(iii) *Let $G = \{g_1, \ldots, g_t\}$ be a set of nonzero polynomials of I with $lc(g_i) \in R^*$ for each $1 \leq i \leq t$. Then, G is a Gröbner basis of I if and only if given $0 \neq r \in I$ there exists an i such that $lm(g_i)$ divides $lm(r)$.*

Remark 2.12.

- The Gröbner theory of skew PBW extensions and some of its important applications in homological algebra have been implemented in Maple in **Fajardo, W.**, *Extended modules and skew PBW extensions*, Ph.D. Thesis, Universidad Nacional de Colombia, Bogotá, 2018 ([4]) and **Fajardo, W.**, *A computational Maple library for skew PBW extensions*, Fund. Inform., 176, 2019, 159–191 ([5], see also [10]).

- The implementation is based on the library `SPBWE.lib` specialized for working with bijective skew PBW extensions. The library has utilities to calculate Gröbner bases, and it includes some functions that compute the module of syzygies, free resolutions and left inverses of matrices, among other things. For the implementation was assumed that $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a bijective skew PBW extension of an LGS ring R and $\mathrm{Mon}(A)$ is endowed with some monomial order $\succeq$.

- **From now on in this work we will assume that $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a bijective skew PBW extension of R, where R is a left noetherian domain**. This implies that A is a left noetherian domain (see [10], Proposition 3.2.1 and Theorem 3.1.5: *Hilbert's basis theorem for skew PBW extensions*). In the examples where we use the library `SPBWE.lib` we have assumed additionally that R is LGS. This implies that A is LGS (see [10], Chapter 15).

3 Algebraic sets and ideals of points for skew PBW extensions

- This last section represents the novelty of the present work. We will study for the skew PBW extensions the algebraic sets, the ideal of points and the relationship between them. Some properties of the affine algebraic sets of commutative algebraic geometry will be extended.

- In particular, we will prove a result (Theorem 3.8) about an adapted version of the classical Hilbert's Nullstellensatz theorem of the commutative algebraic geometry, for **quasi-commutativ**e skew PBW extensions **over algebraically closed fields** and **with polynomial center**. This result covers the classical one.

3.1 Roots of polynomials

For $n \geq 1$, let R^n be the left R-module of vectors over R of n components. Let $f \in A$ and $Z := (z_1, \ldots, z_n) \in R^n$. By Proposition 2.9,there exist polynomials $q_1, \ldots, q_t, h \in A$, with remainder h reduced w.r.t. $F := \{x_1 - z_1, \ldots, x_n - z_n\}$, such that $f \xrightarrow{F}_+ h$ and

$$f = q_1(x_1 - z_1) + \cdots + q_n(x_n - z_n) + h.$$

In general, h is not unique, **and even worse, it could not belong to** R, as the next example shows.

Example 3.1. Consider the Witten algebra (see [10], Chapter 2) $A := \sigma(\mathbb{Q})\langle x, y, z\rangle$ defined by

$$zx = xz - x,\ zy = yz + 2y,\ yx = 2xy.$$

For $Z := (1, -2, -3) \in \mathbb{Q}^3$ and $f := x^2y + xz + yz \in A$, with `SPBWE.lib` the Algorithm Division produces

$$f = (\tfrac{1}{2}xy + \tfrac{1}{4}y)(x - 1) + \tfrac{1}{4}(y + 2) + 0(z + 3) + xz + yz - \tfrac{1}{2},$$

i.e., $q_1 = \frac{1}{2}xy + \frac{1}{4}y$, $q_2 := \frac{1}{4}$, $q_3 = 0$ and $h = xz + yz - \frac{1}{2}$.

Even for quasi-commutative skew PBW extensions the situation is similar. In fact, consider a 3-multiparametric quantum space (see [10], Chapter 4) $A := \sigma(\mathbb{C})\langle x, y, z\rangle$ defined by

$$yx = 2ixy,\ zx = 3ixz,\ zy = -iyz.$$

For $Z := (i, 2i, 3i) \in \mathbb{C}^3$ and $f := x^2y + yz^2 + xz \in A$, with `SPBWE.lib` we found that

$$f = (\tfrac{1}{2}ixy - \tfrac{1}{4}iy)(x - i) + \tfrac{1}{4}(y - 2i) + 0(z - 3i) + yz^2 + xz + \tfrac{1}{2}i,$$

i.e., $q_1 = \frac{1}{2}ixy - \frac{1}{4}iy$, $q_2 := \frac{1}{4}$, $q_3 = 0$ and $h = yz^2 + xz + \frac{1}{2}i$.

Thus, the evaluation of a polynomial $f \in A$ in a given $Z \in R^n$ defined as the remainder in the Division Algorithm is not a good idea. However, the following definition does not depend on the Division Algorithm.

Recall that $\boldsymbol{A := \sigma(R)\langle x_1, \ldots, x_n\rangle}$ is a bijective skew $\boldsymbol{PBW}$ extension of $\boldsymbol{R}$, where $\boldsymbol{R}$ is a left noetherian domain.

Definition 3.2. *Let $n \geq 1$, $f \in A$ and $Z := (z_1, \ldots, z_n) \in R^n$. Z is a **root** of f if and only if f is in the two-sided ideal generated by $x_1 - z_1, \ldots, x_n - z_n$. This condition is denoted by $f(Z) = 0$.*

Thus,

$$f(Z) = 0 \text{ if and only if } f \in \langle Z\rangle,$$

where the two-sided ideal generated by $x_1 - z_1, \ldots, x_n - z_n$ is simply denoted by $\langle Z\rangle$, i.e.,

$$\langle Z\rangle := \langle x_1 - z_1, \ldots, x_n - z_n\rangle. \tag{3.1}$$

Definition 3.3. *Let $f \in A$. The **vanishing set** of f, also called the **set of roots** of f, is denoted by $V(f)$, and defined by*

$$V(f) := \{Z \in R^n \mid f(Z) = 0\}. \tag{3.2}$$

If $S \subseteq A$, then

$$V(S) := \{Z \in R^n \mid f(Z) = 0, \text{ for every } f \in S\}. \tag{3.3}$$

*A subset $X \subseteq R^n$ is **algebraic** if either $X = R^n$ or there exists $g \neq 0 \in A$ such that $X \subseteq V(g)$.*

3.2 Algebraic sets and ideals of points

Some classical properties of affine algebraic sets of commutative algebraic geometry are extended next.

Theorem 3.4. (i) *Let $f, g, h \in A$ and $Z := (z_1, \ldots, z_n) \in R^n$.*

(a) *If $f(Z) = 0 = g(Z)$, then $(f+g)(Z) = 0$.*

(b) *$V(f) \subseteq V(gfh)$.*

(ii) *Let $I := Ag$ be a left principal ideal of A. Then, $V(I) = V(g)$. The same is true for right and two-sided principal ideals of A.*

(iii) (a) *$V(0) = R^n$.*

(b) *$\emptyset$ is algebraic.*

(c) *If $S \subseteq T \subseteq A$, then $V(T) \subseteq V(S)$.*

(d) *If $S \subseteq A$, then $V(S) = V(AS) = V(SA) = V(ASA)$.*

(e) *$V(I) \cup V(J) \subseteq V(I \cap J)$, where I, J are left (right, two-sided) ideals of A.*

(f) *$V(\sum_{k\in\mathcal{K}} I_k) = \bigcap_{k\in\mathcal{K}} V(I_k)$, where I_k is a left left (right, two-sided) ideal of A.*

(h) *Let $Z := (z_1, \ldots, z_n) \in R^n$. Then, $\{Z\} \subseteq V(\langle Z\rangle)$.*

(iv) *Let $X \subseteq R^n$. Then,*

$$I(X) := \{g \in A \mid g(Z) = 0, \text{ for every } Z \in X\}$$

is a two-sided ideal of A, called the ***ideal of points*** *of X. Some properties of $I(X)$ are:*

(a) $I(\emptyset) = A$.

(b) *For $X, Y \subseteq R^n$, $X \subseteq Y \Rightarrow I(Y) \subseteq I(X)$.*

(c) *If I is a left (right, two-sided) ideal of A, then $I \subseteq I(V(I))$.*

(d) $X \subseteq V(I(X))$.

(e) *If $g \in A$, $V(I(V(g))) = V(g)$. Thus, if $X = V(g)$, then $V(I(X)) = X$.*

(f) $I\left(V\left(I\left(X\right)\right)\right) = I\left(X\right)$.

(g) $I(\bigcup_{k\in\mathcal{K}} X_k) = \bigcap_{k\in\mathcal{K}} I(X_k)$.

(h) *Let $Z := (z_1, \ldots, z_n) \in R^n$. Then, $I(\{Z\}) = \langle Z \rangle$.*

Corollary 3.5. (i) *R^n has a* ***Zariski topology*** *where the closed sets are the algebraic sets.*

(ii) *If $X \subseteq R^n$ is finite, then X is algebraic, and hence, closed.*

Remark 3.6. There exist skew PBW extensions such that $V(A) \neq \emptyset$. In fact, let $A := \sigma(\mathbb{Q})\langle x, y, z\rangle$ defined by

$$yx = xy - 1,\ zx = xz,\ zy = yz.$$

Consider the left ideal $I := A(x-1) + Ay + Az$ and observe that

$$1 = -y(x-1) + (x-1)y + 0z = -y(x-1) + (x-1)(y-0) + 0(z-0),$$

i.e., $I = A$ and $(1, 0, 0) \in V(1) = V(A)$.

3.3 Hilbert's Nullstellensatz theorem for skew PBW extensions

Next we prove the main result, namely, to give an adaptation of the Hilbert's Nullstellensatz theorem for quasi-commutative bijective skew PBW extensions such that it covers the classical theorem of commutative algebraic geometry. **For this purpose we need the following preliminary lemma whose proof is supported in the Gröbner theory of skew PBW extensions**.

Lemma 3.7. *Let $A := \sigma(\mathbb{F})\langle x_1, \dots, x_n\rangle$ be a quasi-commutative bijective skew PBW extension of $\mathbb{F}$, where $\mathbb{F}$ is a field. Then, for every $Z := (z_1, \dots, z_n) \in \mathbb{F}^n$, $\langle Z\rangle$ is completely semiprime.*

Theorem 3.8 (Hilbert's Nullstellensatz). *Let $A := \sigma(\mathbb{F})\langle x_1, \ldots, x_n\rangle$ be a quasi-commutative bijective skew PBW extension of $\mathbb{F}$, where $\mathbb{F}$ is an algebraically closed field. Assume that $Z(A)$ is a polynomial ring in n variables with coefficients in $\mathbb{F}$. Let I be a two-sided ideal of A. Then,*

$$\langle I_{Z(A)}(V_{Z(A)}(J))\rangle \subseteq \sqrt{I} \subseteq I(V(I)),$$

where $J := l^{-1}(I)$, $l : Z(A) \to A$ is the inclusion of the center of A in A, $V_{Z(A)}(J)$ is the vanishing set of J with respect to $Z(A)$ and $I_{Z(A)}(V_{Z(A)}(J))$ is the ideal of points of $V_{Z(A)}(J)$ with respect to $Z(A)$.

Proof. $\sqrt{I} \subseteq I(V(I))$:

- If $V(I) = \emptyset$, then $I(V(I)) = A$.
- Assume that $V(I) \neq \emptyset$ and let $f \in \sqrt{I}$, then f is I-strongly nilpotent, and hence, I-nilpotent, so there exists $m \geq 1$ such that $f^m \in I$. Let $Z := (z_1, \ldots, z_n) \in V(I)$, then $f^m \in \langle Z \rangle$. From Lemma 3.7, $f \in \langle Z \rangle$, i.e., $f \in I(V(I))$.

$\langle I_{Z(A)}(V_{Z(A)}(J)) \rangle \subseteq \sqrt{I}$:

- Consider the inclusion $Z(A) \xrightarrow{l} A$ and let $J := l^{-1}(I)$. Then, $J = l(J) = l(l^{-1}(I)) \subseteq I$, and let $\langle J \rangle := AJA$ be the two-sided ideal of A generated by J. We have $\langle J \rangle \subseteq I$, so $\sqrt{\langle J \rangle} \subseteq \sqrt{I}$.
- But $\langle \sqrt{J} \rangle \subseteq \sqrt{\langle J \rangle}$, where $\sqrt{J}$ is the radical of J in the ring $Z(A)$. In fact, let $w \in \sqrt{J}$, then there exists $m \geq 1$ such that $w^m \in J \subseteq \langle J \rangle$, but $w \in Z(A)$, then w is $\langle J \rangle$-strongly nilpotent, i.e., $w \in \sqrt{\langle J \rangle}$.
- Thus, $\langle \sqrt{J} \rangle \subseteq \sqrt{I}$. Applying the classical Hilbert's Nullstellensatz for $Z(A)$ (here we use that $\mathbb{F}$ is algeraically closed) we have $\sqrt{J} = I_{Z(A)}(V_{Z(A)}(J))$, so we get that $\langle I_{Z(A)}(V_{Z(A)}(J)) \rangle \subseteq \sqrt{I}$.

□

Example 3.9. Next we present some concrete examples of skew PBW extensions that satisfy the hypothesis of Theorem 3.8. $\mathbb{F}$ denotes an algebraically closed field.

(i) It is clear that if $A = \mathbb{F}[x_1, \ldots, x_n]$ is the classical commutative polynomial ring and I is an ideal of A, then $Z(A) = A$ and in Theorem 3.8 we have $\langle I_{Z(A)}(V_{Z(A)}(J))\rangle = I(V(I))$, and hence, $I(V(I)) = \sqrt{I}$.

(ii) If $q \neq 1$ is an arbitrary root of unity of degree $m \geq 2$, then the center of the quantum plane $A := \mathbb{F}_q[x, y]$ is the subalgebra generated by x^m and y^m, i.e., $Z(\mathbb{F}_q[x, y]) = \mathbb{F}[x^m, y^m]$, see **Shirikov, E.N.**, *Two-generated graded algebras*, Algebra Discrete Math., 3, 2005, 64–80 ([15]). Recall that the rule of multiplication in A is given by $yx = qxy$.

(iii) The previous example can be generalized in the following way (see [3], Lemma 4.1: **Ceken, S., Palmieri, J., Wang, Y.-H., and Zhang, J.J.**, *The discriminant controls automorphism groups of noncommutative algebras*, Adv. Math., 269, 2015, 551–584.):

Let $q \in \mathbb{F} - \{0\}$ and $A := \mathbb{F}_q[x_1, \ldots, x_n]$ be the skew PBW extension defined by $x_j x_i = q x_i x_j$ for all $1 \leq i < j \leq n$. If $n \geq 2$ and $q \neq 1$ is a root of unity of degree $m \geq 2$, then

(a) If $q = -1$, then

$$Z(A) = \mathbb{F}[x_1^2, \ldots, x_n^2] \text{ when } n \text{ is even.}$$

(b) If $q \neq -1$, then

$$Z(A) = \mathbb{F}[x_1^m, \ldots, x_n^m] \text{ when } n \text{ is even.}$$

THANKS

References

[1] **Birkenmeier, G.F., Kim, J. Y., and Park, J. K.**, *A connection between weak regularity and the simplicity of prime factor rings*, Proc. Amer. Math. Soc., 122, 1994, 53-58.

[2] **Chan, K., Young, A., and Zhang, J.J.**, *The discriminant formulas and applications*, Algebra Number Theory, 10, 2016, 557–596.

[3] **Ceken, S., Palmieri, J., Wang, Y.-H., and Zhang, J.J.**, *The discriminant controls automorphism groups of noncommutative algebras*, Adv. Math., 269, 2015, 551–584.

[4] **Fajardo, W.**, *Extended modules and skew PBW extensions*, Ph.D. Thesis, Universidad Nacional de Colombia, Bogotá, 2018.

[5] **Fajardo, W.**, *A computational Maple library for skew PBW extensions*, Fund. Inform., 176, 2019, 159–191.

[6] **Fulton, W.**, *Algebraic Curves*, Benjamin, 1969.

[7] **Gluesing-Luerssen, H.**, *Introduction to skew-polynomial rings and skew-cyclic codes*, arXiv:1902.03516v2.

[8] **Lezama, O. and Gallego, C.**, *Gröbner bases for ideals of sigma-PBW extensions*, Communications in Algebra, 39 (1), 2011, 50-75.

[9] **Lezama, O. & Reyes, M.**, *Some homological properties of skew PBW extensions*, Comm. in Algebra, 42, 2014, 1200-1230.

[10] **Fajardo, W., Gallego, C., Lezama, O., Reyes, A., Suárez, H., and Venegas, H.**, *Skew PBW Extensions: Ring and module theoretic properties, matrix and Gröbner methods, applications*, Algebra and Applications 28, Springer, 2020.

[11] **McConnell, J. and Robson, J.**, *Noncommutative Noetherian Rings*, Graduate Studies in Mathematics, AMS, 2001.

[12] **McConnell, J. and Robson, J.,** *The Nullstellensatz and generic flatness*, Perspectives in Ring Theory, 1988, pp. 227-232.

[13] **Ore, O.**, *Theory of non-commutative polynomials*. Ann. Math., 34, 1933, 480-508.

[14] **Reyes, A. and Hernández-Mogollón, R.**, *A survey on some characterizations of Hilbert's Nullstellensatz for some non-commutative rings of polynomial type*, Ing. Cienc., 16, 2020, 27-52.

[15] **Shirikov, E.N.**, *Two-generated graded algebras*, Algebra Discrete Math., 3, 2005, 64–80.