THE MOST NON-ALGEBRAIC COMPLEX TORI - AN ALGEBRAIC CONSTRUCTION

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This is a report on a joint work with Tatiana Bandman (Bar-Ilan).

Definition. Let $\mathbb{Q}$ be the field of rational numbers and $g \geq 2$ an integer. We say that a a degree $2g$ polynomial $f(x) \in \mathbb{Q}[x]$ *without real roots* is *very irreducible* if it enjoys the following property.

The polynomial $f(x)$ is irreducible over $\mathbb{Q}$ and its Galois group over $\mathbb{Q}$ (which is apriori transitive) is doubly transitive.

Examples. It follows from results of I. Schur that the *truncated exponent*

$$\exp_{2g}(x) = \sum_{j=1}^{2g} \frac{x^j}{j!}$$

is very irreducible for all g. It follows from results of E. Selmer, E. Nart and N. Vila that if g is *not* congruent to 1 modulo 3 then the polynomial $x^{2g} + x + 1$ is very irreducible.

One may easily check that if $f(x)$ is a degree $2g$ very irreducible polynomial then the quotient $K_f := \mathbb{Q}[x]/f(x)\mathbb{Q}[x]$ is a purely imaginary number field of degree $2g$ that does not contain proper subfields except $\mathbb{Q}$. This implies that the only roots of unity in K_f are 1 and -1.

For each very irreducible $f(x)$ we construct a *simple* g-dimensional complex torus T_f that enjoys the following properties.

(i) The endomorphism algebra of T_f is isomorphic to K_f.
(ii) The Picard number of T_f is 0 and therefore the algebraic dimension of T is 0. In particular, T_f is *not* an abelian variety.
(iii) If $T_f^\vee$ is the dual of T_f then $\mathrm{Hom}(T_f, T_f^\vee) = \{0\}$.
(iv) The automorphism group $\mathrm{Aut}(T)$ of the complex Lie group T is isomorphic to $\mathbb{Z}^{g-1} \times \{\pm 1\}$.

These properties suggest that one may view T_f as transcendental counterparts of abelian varieties of CM type. (Notice that properties (ii)-(iv) follow from property (i) combined with simplicity of T_f.)

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For any given $g \geq 2$ we construct explicitly infinitely many very irreducible polynomials $f_n(x)$ of degree $2g$ with mutually non-isomorphic numbers fields K_{f_n}. This implies that the corresponding g-dimensional simple complex tori T_{f_n} are mutually non-isogenous.

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