Automorphism groups of rigid affine surfaces: the identity component

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Completions of affine surfaces

Let Y be a normal affine algebraic surface over an algebraically closed field $\mathbb{K}$ of characteristic zero.

We prove that $\mathrm{Aut}^\circ(Y)$ either equals an algebraic torus or contains an infinite-dimensional unipotent subgroup $\varinjlim_n \mathbb{G}_a^n$.

Definition

An NC-completion of Y is a completion (X, D) with a normal projective surface X and a normal crossing boundary divisor D contained in a smooth part of X.

$\mathrm{Bir}(X, D)$ is a group of birational maps of X to itself, which send Y to itself isomorphically. Naturally, $\mathrm{Aut}(Y) \cong \mathrm{Bir}(X, D)$.

We provide a condition when $\mathrm{Bir}(X, D)$ acts regularly on X, hence is algebraic. We also describe, when Y admits a unique in combinatorial sense minimal model, i.e., an NC-completion without exceptional curves.

Resolving indeterminacies

A (−1)-curve is a rational smooth closed curve in X with self-intersection index -1. A blowup of X at a smooth point p is an inverse of a morphism $\mathrm{Bl}_p(X) \to X$ that sends a (-1)-curve to p.

Resolving indeterminacies of a birational map of NC-completions $f\colon (X_1, D_1) \dashrightarrow (X_2, D_2)$ and of f^{-1}, we get the diagram:

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{D}) & \xrightarrow[\iota]{\simeq} & (\widetilde{X}, \widetilde{D}) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ (X_1, D_1) & \underset{f}{\dashrightarrow} & (X_2, D_2) \end{array}$$

Here p_i are sequences of blowdowns at the boundary and ι is an isomorphism.

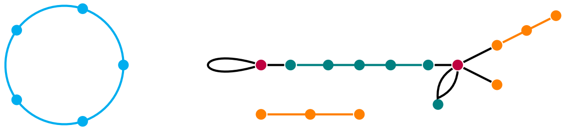
We may assume that no (-1)-curve in $(\widetilde{X}, \widetilde{D})$ is contracted by both p_1 and p_2, i.e., the diagram is relatively minimal.

Dual graphs

A weighted graph $\Gamma(D)$ is constructed from (X, D) by sending each irreducible curve C in D to a vertex with weight C^2, each node (intersection point) to an edge.

A vertex is branching if its degree is ≥ 3 or the corresponding curve is non-rational.

Segments are connected components of Γ without the set of branching vertices $\mathrm{Br}(\Gamma)$. A segment can be circular, inner linear, or extremal linear.



Graph Lemma

A segment is called admissible if all vertices are of weight ≤ -2. A blowup at the node of the boundary divisor is called inner and outer elsewhere. An inner transformation is a sequence of inner blowups and inner blowdowns.

Lemma

A relatively minimal diagram between minimal NC-pairs (X_1, D_1) and (X_2, D_2) induces a transformation from $\Gamma(D_1)$ to $\Gamma(D_2)$, which includes

1. an isomorphism between $\mathrm{Br}(\Gamma(D_1))$ and $\mathrm{Br}(\Gamma(D_2))$;
2. a one-to-one correspondence between segments of $\Gamma(D_1)$ and ones of $\Gamma(D_2)$;
3. isomorphisms on admissible segments;
4. inner transformations on inner (non-admissible) segments.

Corollary

Any $f \in \mathrm{Bir}(X, D)$ is represented by an inner transformation if and only if $\Gamma(D)$ contains no non-admissible extremal segment.

Rigid surfaces

Proposition

Given an affine surface Y and a minimal NC-completion (X, D), the following are equivalent:

1. Y admits an effective $\mathbb{G}_\mathrm{a}$-action;
2. Y admits an $\mathbb{A}^1$-fibration over a smooth affine curve;
3. the dual graph $\Gamma(D)$ has a (0)-vertex of degree one;
4. the dual graph $\Gamma(D)$ has a non-admissible extremal linear segment.

Theorem (P.–Zaidenberg)

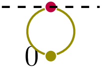
1. If all segments of $\Gamma(D)$ are admissible, then $\mathrm{Aut}(Y)$ is an algebraic group acting regularly on X;
2. If all extremal segments of $\Gamma(D)$ are admissible, then $\mathrm{Aut}^\circ(Y)$ is an algebraic torus;
3. If $\Gamma(D)$ contains a non-admissible extremal segment, then $\mathrm{Aut}^\circ(Y)$ contains an infinite-dimensional abelian unipotent subgroup.

Uniqueness of a minimal model

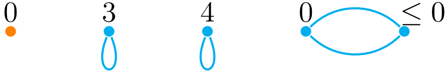
If all segments of an (abstract) weighted graph Γ are admissible, then it admits a unique minimal model. The converse is not always true:

Proposition (P.–Zaidenberg)

1. If all segments of Γ are either admissible or charm earrings, then it admits a unique minimal model;



2. If Γ coincides with one of the following graphs, then again it admits a unique minimal model;



3. Otherwise Γ admits an infinite number of minimal models.

Thank You!