Matrix Waring Problem

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The classical Waring problem in number theory asks if, for a given positive integer k, there exists $g(k)$, such that every natural number is a sum of $g(k)$ many k-th powers, and $g(k)$ is smallest with this property. That is, the function $X_1^k + X_2^k + \cdots + X_{g(k)}^k$ represents all natural numbers. This problem has a long history starting from Waring in 1770. Waring problem has also been considered for various other rings, such as the ring of integers, polynomial rings, etc. Modern versions consider the analogues question over objects with noncommutative structures, such as groups, Lie algebras, matrix algebras, etc. Various generalisation of this problem considers more general words, instead of just power words, and at times ignore small size for better bounds. For example, Shalev [6] showed that for every finite (nonabelian) simple group of sufficiently high order every element can be expressed as values of word w of length 3. This was later improved to 2 by Larsen, Shalev and Tiep [3].

Larsen conjectured that a similar result should hold for matrices over finite fields. In other words, if R denotes a commutative ring with unity, then the Matrix Waring Problem would be to address whether matrices over R, possibly when the entries are "large" enough, can be expressed as a sum of two kth powers (of matrices). The goal of this article is to answer this question in the case where R is a finite field $\mathbb{F}_q$, with q sufficiently large. We have, for all integers $k \geq 1$, there exists a constant C_k depending only on k, such that for all $q > C_k$ and for all $k \geq 1$ every matrix in $M_n(\mathbb{F}_q)$ is a sum of two k-th powers.

This work is done in collaboration with Krishna Kishore. In [1], he proved that it can be written as a sum of at most three k-th powers which we have improved to 2 now. The key idea is to use powers in $GL(n, q)$ from [2], and Lang-Weil's results on the number of solutions to equations over finite fields [4, 5].

References

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