Algebraic Group Theory and Model Theory: Some Surprising Links

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Algebraic Groups, Their Friends and Relations
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Model Theory

- Studies logic properties of mathematical objects and mathematical theories.
- Classifies mathematical objects within the same category (say, groups, or rings, etc.) up to elementary equivalence.
- Objects from different categories are studied in respect of their interpretability of one of them in another, or bi-interpretability; for example, **algebraic groups (and their friends and relations)** which have sufficiently stringent structural properties are bi-interpretable with their base fields—or rings.

A brilliant example of that was just given by Elena Bunina.

Groups of finite Morley rank

- Morley rank: introduced by Morley in the 1960s in a rather general setup.
- Groups of finite Morley rank (fMr): abstract groups equipped with a suitable notion of dimension—Morley rank—on definable sets.
- Morley rank behaves very much as dimension o constructive sets in algebraic geometry.
- Actually, algebraic groups over a.c. fields are groups of fMr, rank being the dimension.
- There are groups of fMr which are not algebraic.

Siblings of algebraic groups

The Cherlin-Zilber Conjecture, circa 1980:

Simple groups of fMr are simple algebraic groups over a.c. fields.

The Conjecture is still open, but a considerable progress was achieved by imitating

Classification of Finite Simple Groups (CSFG).

This is less surprising than one may think, because

The classification of simple algebraic groups over a.c. fields
is an easy model-theoretic corollary
of CFSG.

Where did groups of finite Morley rank come from?

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No.

In the 1970s they became needed in the role of **binding groups**, introduced by Zilber as model-theoretic analogues of Galois groups.

They **act** on some important structures appearing in Model Theory.

They are supposed to be studied as **permutation groups**.

Applications of CSFG

The range of applications of CFSG is huge.
Look at CFSG: what is needed in applications, is, as a rule, information about **actions** of finite groups.,

- Combinatorics. First of all classification of 2-transitive groups — it was impossible to get without CSFG, cf. talk by Gareth Jones.
- Aslo Representation Theory, Model Theory...

The first signs of a similar process appeared in model theory only recently:
Freitag and Moosa 2021. *Bounding nonminimality and a conjecture of Borovik-Cherlin.*

Why this delay?

Because the Cherlin-Zilber Conjecture is still open.

Confirmed in the case when a simple group G of fMr contains an infinite elementary abelian 2-subgroup (G is of **even** type): in this case G is a simple algebraic group over a.c. field **of characteristic** 2.

Proof: **Altınel, B, Cherlin 2008**: $\approx$ 400 pages in a book of 525 pages.

Non-even types: A huge lot is done, but this is still not enough. However, these results frequently provide, in business speak, **actionable information** which could be used in the study of permutations groups of fMr.

Definable actions of groups of fMr

In this talk:

- G: a group of fMr, usually connected
- $\mathrm{rk}\, G$: Morley rank of G
- X: set on which G acts definably, $X \times G \longrightarrow X$
- Action: faithful, transitive — usual group-theoretic meaning
- Action is **imprimitive**, if there is a non-trivial **definable** G-invariant equivalence relation E on X, and **primitive** otherwise.
- Equivalently: action is primitive if it is transitive and a stabiliser of a point is a maximal **definable** subgroup in G.

First difficulties in the study of permutation groups of fMr

A finite permutation group on a set on n elements has order $\leqslant n!$

Unfortunately, the Morley rank of G acting faithfully on X with $\operatorname{rk} X = n$ is not bounded even in case of $n = 2$.

This happens already in the case of algebraic groups.

Example

K an a. c. field and $A \simeq K^n$.

$T \simeq K^\times$ acts on A by matrices $\mathrm{diag}(t, t^2, \dots, t^n)$ in a fixed basis.

B a hyperplane in A with $\bigcap_{t \in T} B^t = 0$.

The right coset action of $G = A \rtimes T$ on $X = G/B$ is **faithful and transitive**.

$\mathrm{rk}\, G = n + 1$ and $\mathrm{rk}\, X = 2$.

However, since $B < A$, the stabiliser of the point B is not maximal in G; hence this action is **imprimitive**.

Some history: Évarist Galois

The concept of primitive groups was introduced by Galois.

- $f(x) \in F[x]$ is a polynomial irreducible in $F[x]$
- K is a the extension of F by all roots $\alpha_1, \dots, \alpha_n$ of $f(x)$

If the Galois group G of $f(x)$ acts imprimitively on $\{\alpha_1, \dots, \alpha_n\}$,
then $f(x)$ factorises as

$$f = f_1 \cdots f_m, \qquad m \geqslant 2,$$

with $f_i \in L[x]$ for some intermediate field $F < L < K$.

P. M. Neumann, *The Concept of Primitivity in Group Theory and the Second Memoir of Galois*. Arch. Hist. Exact Sci. 60, no. 4 (July 2006), 379–429.

An almost trivial observation

If

G is transitive and definably imprimitive on X and preserves a non-trivial definable equivalence relation E

then

G has a transitive definable action on a "smaller" factor set X/E.

Therefore imprimitivity of the action sometimes allows us to apply inductive arguments.

A bound for definably primitive action

B-Cherlin 2008: There exists a function $f : \mathbb{N} \to \mathbb{N}$ with the following property.

If a group G of finite Morley rank acts on a set X definably, transitively and definably primitively, then:

$$\mathrm{rk}(G) \leq f(\mathrm{rk}\, X).$$

Proof (50+ pages) uses the full strength of classification results on group of fMr of even and odd types known by that time.

Unfortunately, we have no classification of simple groups of fMr; instead of a **single theorem**, we have to refer to a **theory** made of many theorems.

It is not clear whether it could be significantly shortened with the current knowledge. However, some progress was made towards an explicit bound – more about that later.

Macpherson and Pillay: The O'Nan-Scott analysis

Setup:

- G is a definable connected primitive permutation group of fMr on X.
- $H < G$ is the stabiliser of a point in X, so that X can be identified with the factor set G/H.

Then G is a permutation group of one of the types 1, 2, 3, 4 described below.

Macpherson and Pillay: The O'Nan-Scott analysis

1. *Affine groups.* Here, G is a semidirect product $G = V \rtimes H$, where V is either elementary abelian or divisible torsion-free abelian, and H acts on V faithfully, and V does not have any definable H-invariant subgroups other than 0 and V.
2. *Unique non-abelian simple regular normal subgroup.* Here G is a semidirect product $G = L \rtimes H$, with L simple and H acting on L faithfully and without leaving invariant any non-trivial proper definable subgroup of L.
3. *Almost simple groups.* Here $L \lhd G$ is definable and simple, $C_G(L) = 1$, and H is a maximal definable subgroup of G; unlike the Case 2, $H \cap L \neq 1$.
4. *Simple diagonal action.* G is a direct product $M_1 \times M_2$ of two isomorphic *monolithic* groups, that is, groups which have a unique minimal definable normal subgroup, and this subgroup is simple; H is the diagonal subgroup of the direct product.

Further directions, I

Definably primitive groups of fMr of classes 1 and 2:

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are open to analysis, and they provide a good testing ground for a systematic application of the highly developed classification machinery of groups of finite Morley rank to questions in model theory.

A historic remark on the affine primitive groups

They appeared in the celebrated theorem by Galois:

> *A solvable primitive permutation group has degree p^k for some prime number p.*

This was the reason why Galois constructed Galois fields.

Further directions, II

However classification of classes 3 and 4

3. *Almost simple groups.* Here $L \lhd G$ is definable and simple, $C_G(L) = 1$, and H is a maximal definable subgroup of G; unlike the Case 2, $H \cap L \neq 1$.
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is obviously equivalent to the Cherlin-Zilber Conjecture.

But it is groups from classes 1 and 2 that are more likely to appear in important configurations in Model Theory.

Multiply generically transitive actions

Definition. *A connected group of finite Morley rank G acting definably on a set X of Morley degree 1 is* **generically n-transitive** *on X if G has a generic orbit on $X^n = X \times \cdots \times X$. The maximal such n is called the* **degree** *of generic transitivity of G on X.*

Examples.

- The general linear group $\mathrm{GL_n}(K)$ is generically n-transitive in its action on the vector space K^n.
- The affine group $\mathrm{GA}_n(K) = K^n \rtimes \mathrm{GL}_n(K)$ is generically $(n+1)$-transitive in its natural action on the affine space K^n.
- The projective group $\mathrm{PGL}_{n+1}(K)$ is generically $(n+2)$-transitive on the n-dimensional projective space $\mathrm{P}^n(K)$.

Borovik-Cherlin Conjecture

As mentioned by **Freitag and Moosa 2005**:

If G acts generically $n+2$-transitively on X, $\operatorname{rk} x = n$, then this action is equivalent to $G = PGL_{(n+1)}(K)$ in its natural action on $X = \mathbb{P}^n(K)$ for some a.c. field K.

Freitag and Moosa proved it for some special classes of structures.

Multiply generically transitive actions, algebraic groups

For algebraic groups and rational actions, the classification has been obtained only in zero characteristic (however, Vladimir Popov suggested that in non-zero characteristics it is not much more complex).

Popov 2005:

Let G be a simple algebraic group over an a. c. field of characteristic 0. Then the maximal degree of generic transitivity for an action of G on an irreducible algebraic variety is as follows:

A_n	$B_n, n \geqslant 3$	$C_n, n \geqslant 2$	$D_n, n \geqslant 4$	E_6	E_7	E_8	F_4	G_2
$n+2$	3	3	3	4	3	2	2	2

Bounds on Morley ranks of primitive groups

B-Cherlin 2008:
Let G be a definably primitive permutation group on X, $\operatorname{rk} X = n$.
There are functions $\rho, \tau : \mathbb{N} \longrightarrow \mathbb{N}$ such that

- The degree t of generic transitivity of G on X is bound by $\tau(n)$.
- The Morley rank of G is bound by $\rho(n)$ which satisfies

$$n\tau(n) \leqslant \rho(n) \leqslant n\tau(n) + \binom{n}{2}.$$

Bounds in the affine case

Berkman-B, 2022: In the affine case, $\tau(n) \leqslant n+1$. In the case of equality,

$$G = K^n \rtimes \mathrm{GL}_n(K)$$

for some a.c. field K and the standard action of $\mathrm{GL}_n(K)$ on K^n.

Corollary. In the affine case,

$$\operatorname{rk} G \leqslant n(n+1) + \binom{n}{2},$$

where $n = \operatorname{rk} X$.

It still can, and should, be improved.

Proof

It uses everything: the whole **theory** of groups of fMr instead of a single unifying classification **Theorem**.

- Ayşe Berkman and I started the project in about 2010.
- Proof is spread over 5 papers:

  Berkman-B 2011, 2012, 2018, 2022, 2023?
- Each of them is completely different from others by the technique used.
- Paper of 2022 uses new results (2020) on actions of finite groups on abelian groups of finite Morley rank.
- The last one deals with the case of groups of even type where simple groups are algebraic – but what about their actions on abelian groups?

Linearisation of actions of simple algebraic groups.

Used in the proof:

Theorem

(B 2020) *Let $G = G_1 \times \cdots \times G_m$ where each G_i is the group of points over some algebraically closed field K_i of characteristic $p > 0$. a simple algebraic group defined over K_i.*

Assume that G acts faithfully, definably and irreducibly on a connected elementary abelian p-group V of finite Morley rank.

Then all K_i are definably isomorphic to the same field K and V has a structure of a finite dimensional K-vector space compatible with the action of G, and G is a Zariski closed subgroup of $GL_K(V)$.

This theorem started as Question B.38 in the book with **Ali Nesin** of 1994.

Surprisingly this appears not to be known...

Corollary

G *is a simple algebraic group over an algebraically closed field* K *of characteristic* $p > 0$

V *a unipotent group of exponent* p

$H = V \rtimes G$ *is also an algebraic group over* K

V *does not have closed* G*-invariant subgroups other than* 0 *and* V.

Then

V has a structure of a finite dimensional vector space over K invariant under the action of G.

Comments on the linearisation

- The model-theoretic assumptions in the Theorem can perhaps be relaxed.
- Required a study of definable actions of **finite** groups on abelian groups of fMr.

Other (not affine) cases

The work only begins, but results are already promising:

Tuna Altınel, Adrien Deloro, Josh Wiscons.

Interestingly, it requires study of actions of alternating groups Alt_n and symmetric groups Sym_n on simple groups of finite Morley rank.

Very hard and technical results.

Better understanding of groups of fMr and **odd** type would help – but here there also promising developments (**Burdges and Cherlin**, **Deloro and Wiscons**).

Many thanks for your attention!

Happy Birthday, Kolya!