Algebraic Group Theory and Model Theory: Some Surprising Links

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Abstract. The talk discusses some interactions and similarities between the theories of algebraic groups over algebraically closed fields and groups of finite Morley rank in model-theoretic algebra.

Introduction

Model theory studies logical properties of mathematical structures, with useful applications to specific classes of structures or theories; group theory and differential algebra provide some examples, among many others. On the other hand, there are groups appearing in the model theory itself for intrinsic reasons. One noticeable example is groups of finite Morley rank. The concept of Morley rank was introduced by Morley in the 1960s in a rather general setup, but groups of finite Morley rank (fMr), abstract groups equipped with a suitable notion of dimension—Morley rank—on definable sets, took a prominent role in Model Theory after they appeared in the role of *binding groups*, some model-theoretic analogues of Galois groups, introduced by Boris Zilber in the 1970s. The groups of points of algebraic groups over algebraically closed fields are groups of fMr, the latter being the Zariski dimension of constructible sets. However, there are non-algebraic groups of fMr.

The Cherlin-Zilber Conjecture proposes that every infinite simple group of fMr is a simple algebraic group. It remains open despite a considerable progress towards its proof, more details, definition, notations are in [1].

The groups of fMr, by their birth as binding groups, are interesting as *permutation groups* (with action assumed to be definable). So we look at permutations groups of fMr.

We denote the Morley rank of a set X as $\operatorname{rk} X$. In all results below, G is assumed to be a connected group of fMr, all actions are definable.

1. Permutation groups of finite Morley rank

1.1. Bounds for the rank and multiple generic transitivity

There are easy examples of groups of fMr of arbitrarily large rank acting faithfully and definably on sets of Morley rank 2. However, the situation is different for primitive actions.

Theorem 1.1. (Borovik-Cherlin [5]) *There exists a function $f : \mathbb{N} \to \mathbb{N}$ with the following property. If G acts on a set X definably, transitively and definably primitively, then:*

$$\mathrm{rk}(G) \leq f(\mathrm{rk}\, X).$$

We will repot later in this paper some progress made towards an explicit bound.

Proof of Theorem 1.1 used reduction of a bound for $\mathrm{rk}\, G$ to a bound for the *degree* of generic transitivity of G on X.

Definition 1.2. *A group G acting definably on a set X of Morley degree* 1 *is said to be* generically n-transitive *on X if G has a generic orbit on $X^n = X \times \cdots \times X$. The maximal such n is called the* degree *of generic transitivity of G on X.*

Examples.

- The general linear group $GL_n(K)$ is generically n-transitive in its action on the vector space K^n.
- The affine group $GA_n(K) = K^n \rtimes GL_n(K)$ is generically $(n+1)$-transitive in its natural action on the affine space K^n.
- The projective group $PGL_{n+1}(K)$ is generically $(n+2)$-transitive on the n-dimensional projective space $P^n(K)$.

For algebraic groups and rational actions, the classification of multiply transitive actions has been obtained only in zero characteristic.

Theorem 1.3. (Popov [7]) *Let G be a simple algebraic group over an algebraically closed field of characteristic* 0*. Then the maximal degree of generic transitivity for an action of G on an irreducible algebraic variety is as follows:*

A_n	$B_n, n \geqslant 3$	$C_n, n \geqslant 2$	$D_n, n \geqslant 4$	E_6	E_7	E_8	F_4	G_2
$n+2$	3	3	3	4	3	2	2	2

1.2. Structure of primitive groups of finite Morley rank

We use the following result which transfers to the realm of groups of fMr the celebrated theorem by O'Nan and Scott in the finite group theory>

Theorem 1.4. (Macpherson and Pillay [6]) *Every primitive group of fMr belongs to one of the following types.*

1. Affine groups. *Here, G is a semidirect product $G = V \rtimes H$, where V is either elementary abelian or divisible torsion-free abelian, and H acts on V faithfully, and V does not have any definable H-invariant subgroups other than* 0 *and V.*

2. Unique non-abelian simple regular normal subgroup. *Here G is a semidirect product $G = L \rtimes H$, with L simple and H acting on L faithfully and without leaving invariant any non-trivial proper definable subgroup of L.*
3. Almost simple groups. *Here $L \lhd G$ is definable and simple, $C_G(L) = 1$, and H is a maximal definable subgroup of G; unlike the Case 2, $H \cap L \neq 1$.*
4. Simple diagonal action. *G is a direct product $M_1 \times M_2$ of two isomorphic* monolithic *groups, that is, groups which have a unique minimal definable normal subgroup, and this subgroup is simple; H is the diagonal subgroup of the direct product.*

1.3. Bounds in the affine case

Theorem 1.5. (Berkman and Borovik [2, 3]) *In the affine case, the degree of transitivity of G does not exceed $n+1$. In the case of equality,*

$$G = K^n \rtimes \mathrm{GL}_n(K)$$

for some a.c. field K and the standard action of $\mathrm{GL}_n(K)$ on K^n.

Now it follows from [5] that

Corollary 1.6. *In the affine case,*

$$\mathrm{rk}\, G \leqslant n(n+1) + \binom{n}{2},$$

where $n = \mathrm{rk}\, X$.

We expect a better bound, $\mathrm{rk}\, G \leqslant n(n+1)$, again, with equality reached only for $G = K^n \rtimes \mathrm{GL}_n(K)$.

2. Linearisation of Actions of Simple Algebraic Groups.

The following is used in proof of Theorem 1.5.

Theorem 2.1. (Borovik [4]) *Let $G = G_1 \times \cdots \times G_m$ where each G_i is the group of points over some algebraically closed field K_i of characteristic $p > 0$. A simple algebraic group defined over K_i.*

Assume that G acts faithfully, definably and irreducibly on a connected elementary abelian p-group V of finite Morley rank.

Then all K_i are definably isomorphic to the same field K and V has a structure of a finite dimensional K-vector space compatible with the action of G, and G is a Zariski closed subgroup of $GL_K(V)$.

Surprisingly this special case appears to be not known:

Corollary 2.2. *Let G is a simple algebraic group over an algebraically closed field K of characteristic $p > 0$, V a unipotent group of exponent p, and assume that $H = V \rtimes G$ is also an algebraic group over K. Assume also that G acts on V irreducibly, that is, does not have closed G-invariant subgroups other than 0 and V.*

Then V has a structure of a finite dimensional vector space over K invariant under the action of G.

Final remarks

Proof of Theorem 1.5 is spread over 5 papers, of which [2, 3] only completed the development of a theory which involves almost everything which is known about groups of finite Morley rank, and this is a considerable body of knowledge. Parallels with the finite group theory and with the algebraic group theory saturate the theory of permutation groups of fMR; new connections are likely to be discovered.

References

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