Matrix divisors and Chevalley groups

Oleg Sheinman

Steklov Mathematical Institute
Moscow

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Matrix divisors

A.Weil,1938; A.Grothendieck, 1957; A.Tyurin, 1964-66

Set-up:

Riemann surface Σ, semisimple Lie algebra $\mathfrak{g}/\mathbb{C}$, a faithful $\mathfrak{g}$-module V, the corresponding Chevalley group $G = G((z))$ over $\mathbb{C}((z))$.

($\mathbb{C}((\cdot))$-Laurent expansions, $\mathbb{C}[[\cdot]]$-Taylor expansions), $\mathbb{C}(\cdot)$ - rational functions)

Given a covering $\mathcal{U} = \{U\}$ of Σ, a Chech 0-cochain with coeffs in $G((z))$ is a correspondence $U \to A_U$, $A_U \in G((z))$ being a Laurent expansion centered at a unique point $\gamma \in U$.

Two cochains A and B are *equivalent* ($A \sim B$) if $\exists C$ – a cochain with coeffs in $G[[z]]$ s.t. $A = CB$.

Matrix divisors = coChains/ $\sim$

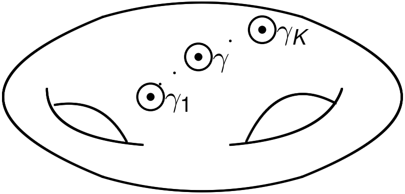
Matrix divisors and holomorphic vector bundles

Given a covering $\Sigma \subset (\bigcup_{\gamma\in\Gamma} U_\gamma) \bigcup U_\infty$, take a Chech 0-cochain $A = \{A_\gamma\ (\gamma \in \Gamma),\ A_\infty\}$ with coeffs. in $G((z))$. Then the cocycle

$$\partial A|_{U_\gamma \cap U_\infty} = A_\gamma A_\infty^{-1}$$

gives a holomorphic vector bundle with the fiber V.

Vice versa, given a holomorphic vector bundle, its gluing functions always split as $g_{\gamma,\infty} = A_\gamma A_\infty^{-1}$ where A_γ can be continued meromorphically into U_γ, and A_∞ to U_∞, thus giving a cochain.



Canonical form of a matrix divisor

z – local parameter on Σ, $h \in L_V^*$

$z^h \in \mathrm{End}(V) : z^h v = z^{\mu(h)} v$ for $v \in V_\mu$ (i.e. z^h – diagonal matrix)

Fact: max torus $T \subset G$ consists of z^h, s.t. $h \in L_V^*$.

Def. The chamber $A_+ \subset T$ is generated by z^h,
$h \in L_V^* \cap$ Weil chamber

THEOREM (ON ELEMENTARY DIVISORS):

1) $G = KA_+K$ where $K = G[[z]]$ (Cartan decomposition);
2) A_+-component of the decomposition is defined up to $T \cap K$.

Corollary: up to equivalence $\Psi_\gamma = z^{h_\gamma} K_\gamma$ ($K_\gamma \in K$).

h_γ, $\gamma \in \Gamma$ – discrete invariants of the matrix divisor,
K_γ, $\gamma \in \Gamma$ give moduli.

Classification problem

Given a divisor Ψ its section is a local meromorphic V-valued function f s.t. Ψf is holomorphic in the domain of definition.

Problem: classify matrix divisors with the same discrete invariants Γ, h and the same sheaf of sections

THEOREM: 1° The moduli space of matrix divisors is a homogeneous space $\mathcal{M}_{\Gamma,h} = \underbrace{K \times \ldots \times K}_{|\Gamma| \text{ times}} / \prod_{\gamma\in\Gamma} K^0_\gamma$ where $K^0_\gamma \subset K$ is the stationary subgroup of the divisor z^{h_γ}.
2° For its tangent space at the unit we have

$$T_e\mathcal{M}_{\Gamma,h} = \bigoplus_{\gamma\in\Gamma,\, \alpha\in R^+ \,:\, \alpha(h_\gamma)>0} \left(\mathbb{C}[[z]]/z^{\alpha(h_\gamma)}\mathbb{C}[[z]]\right) x_\alpha .$$

Dimension of a moduli space

Let $\mathcal{N}_\gamma$ be a number of parameters at a $\gamma \in \Gamma$. It follows by previous theorem that

$$\mathcal{N}_\gamma = \sum_{\alpha \in R^+ : \alpha(h_\gamma) > 0} \alpha(h_\gamma) = \sum_{s>0} s \dim \mathfrak{g}_s^\gamma$$

where $\mathfrak{g}_s = \{X \in \mathfrak{g} \mid (\operatorname{ad} h_\gamma)X = sX\}$. Below we assume h_γ to be dual to a simple root, say α_γ, and $(\alpha : \alpha_\gamma)$ to denote multiplicity of α_γ in a root α. Then

$$\mathcal{N}_\gamma = \sum_{\alpha > 0} (\alpha : \alpha_\gamma).$$

Regard to $\gamma \in \Gamma$ as to free parameters, and identify matrix divisors related by common conjugation by a constant element of G. Denote by $\mathcal{M}_h$ the corresponding moduli space. Then

$$\dim \mathcal{M}_h = \sum_{\gamma \in \Gamma} (1 + \mathcal{N}_\gamma) - \dim \mathfrak{g}.$$

When $\dim \mathcal{M}_h = (\dim \mathfrak{g})(g-1)$?

Assume, h_γ is independent of γ. Then it also holds for $\mathcal{N}_\gamma$:

$$\mathcal{N}_\gamma = \mathcal{N}, \quad \forall \gamma \in \Gamma.$$

It follows that $\dim \mathcal{M}_h = (1+\mathcal{N})|\Gamma| - \dim \mathfrak{g}$, hence we need

$$(1+\mathcal{N})|\Gamma| = (\dim \mathfrak{g})g.$$

Since $\mathfrak{g}$ and the Riemann surf. are absolutely independent, generically $1+\mathcal{N}$ is not divisible by g. Assume $\exists r \in \mathbb{Z}_+$ s.t. $|\Gamma| = rg$. Then

$$(1+\mathcal{N})r = \dim \mathfrak{g}.$$

The only general reason for the last is that

$$r = \operatorname{rank} \mathfrak{g}, \quad \mathcal{N} = \text{Coxeter number}.$$

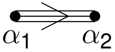
Our question reads now: **When $\mathcal{N}$ = Coxeter number ?**

How to calculate $\mathcal{N}$? (examples)

A_{n-1}:

Positive roots: $\alpha_i + \ldots + \alpha_j$ $(1 \leq i < j \leq n)$.

Roots of height 1 in α_1: $\alpha_1 + \ldots + \alpha_j$ $(1 < j \leq n)$. Hence $\dim \mathfrak{g}_1 = n - 1$, $\mathfrak{g}_2$ is absend. $\boxed{\mathcal{N} = \dim \mathfrak{g}_1 = n - 1}$.

G_2: 

, grading by multiplicity of α_2

Positive roots: $\alpha_1, \underbrace{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2}_{\mathfrak{g}_1}, \underbrace{3\alpha_1 + 2\alpha_2}_{\mathfrak{g}_2}$

$$\dim \mathfrak{g}_1 = 4,\ \dim \mathfrak{g}_2 = 1,\ \boxed{\mathcal{N} = \dim \mathfrak{g}_1 + 2 \dim \mathfrak{g}_2 = 6}.$$

Relation between the number of parameters at a point and the Coxeter number for classical Lie algebras

	$\dim \mathfrak{g}_1$	$\dim \mathfrak{g}_2$	$\dim \mathfrak{g}_1 + 2\dim \mathfrak{g}_2$	Coxeter number
A_{n-1} $(\mathfrak{gl}(n))$	$n-1$	0	$n-1$	–
A_{n-1} $(\mathfrak{sl}(n))$	$n-1$	0	$n-1$	n
B_n	$2n-1$	0	$2n-1$	$2n$
C_n	$2n-2$	1	$2n$	$2n$
D_n	$2n-2$	0	$2n-2$	$2n-2$
G_2	4	1	6	6

Comments: 1) $\mathcal{N} = \dim \mathfrak{g}_1 + \dim \mathfrak{g}_2$ in all lines;
2) h is dual to the shortest terminal root α_1 of the Dynkin diagr.;
3) $\mathcal{N} =$ Coxeter number for $A_{n-1}(\mathfrak{gl}(n))$, C_n, D_n, G_2, and does not hold for $A_{n-1}(\mathfrak{sl}(n))$, B_n.

Conformal extensions

G – a simple Lie group$/\mathbb{C}$, $\mathcal{Z}$ is its center.

$$CG := G \otimes_{\mathcal{Z}} \mathbb{C}^*$$

Let $\mathfrak{g} = Lie(G)$, $\tilde{\mathfrak{g}} = Lie(CG)$, then

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes 1 \oplus 1 \otimes \mathbb{C} \simeq \mathfrak{g} \oplus \mathbb{C}.$$

G	$\mathcal{Z}$
$SL(n)$	$\mathbb{Z}_n$
$SO(2n+1)$	$\mathbb{Z}_2$
$Sp(2n)$	$\mathbb{Z}_2$
$SO(2n)$	$\mathbb{Z}_2$

Grading: h operates on $\tilde{\mathfrak{g}}$ as $\mathrm{ad}\, h \otimes 1$ (on $\mathfrak{g} \otimes 1$ only).

Hence

$$1 \otimes \mathbb{C} \subset \tilde{\mathfrak{g}}_1.$$

Natural CG-module: $V \otimes V_1$ where V is the standard G-module, $V_1 \simeq \mathbb{C}$, $(g \otimes \lambda)(v \otimes v_1) = gv \otimes \lambda^{|\mathcal{Z}|} v_1$.

Relation between the number of parameters at a point and the Coxeter number for conformal extensions

	$\tilde{\mathfrak{g}}_1$	$\dim \tilde{\mathfrak{g}}_1$	$\dim \mathfrak{g}_2$	$\dim \tilde{\mathfrak{g}}_1 + 2 \dim \mathfrak{g}_2$	Coxeter number
A_{n-1}	$\mathfrak{g}_1 \oplus \mathbb{C}$	n	0	n	n
B_n	$\mathfrak{g}_1 \oplus \mathbb{C}$	$2n$	0	$2n$	$2n$
C_n	$\mathfrak{g}_1$	$2n-2$	1	$2n$	$2n$
D_n	$\mathfrak{g}_1$	$2n-2$	0	$2n-2$	$2n-2$
G_2	$\mathfrak{g}_1$	4	1	6	6

Comments: 1) conformal extensions are considered for A_{n-1} and B_n only;
2) By A_{n-1} we mean $\mathfrak{sl}(n)$ here;

Questions

- Do conformal extensions appear as a particular case of some general mathematical construction?
- What could be a general reason for the relation

$$\sum_s s \dim \mathfrak{g}_s = \text{Coxeter number} \quad ?$$

- What do the above considered matrix divisors have in common? Why do we need a conformal extension in some cases, and do not in the others, in order to obtain the "correct" dimension?

  Conjecture (speculation): the above matrix divisors are distinguished by the property that the corresponding holomorphic vector bundles are stable.

Matrix divisors and flag configurations

Recall, given a matrix divisor Ψ, by its section we mean a meromorphic V-valued function f on U s.t. Ψf is holomorphic in a neighborhood of any $\gamma \in \Gamma \cap U$.

For any γ and a local coordinate $z: \; z(\gamma) = 0$ let

$$f(z) = \sum_{i=-k}^{\infty} f_i z^i.$$

Let F_i be the subspace in V constituted by the components f_i of all solutions $(f_{-k}, f_{-k+1}, \dots, f_{m-1})$ to the system $\Psi f = 0$. Then

$$F_{-k} \subseteq F_{-k+1} \subseteq \dots \subseteq F_{m-1} \subseteq V.$$

Flag configuration: the set of such flags F^γ, $\gamma \in \Gamma$

COROLLARY: f is a section iff $f_i \in F_i^\gamma$ for $\forall \gamma \in \Gamma$.

Matrix divisors and ∞-dimensional Lie algebras

DEFINITION: Given a matrix divisor Ψ we call the Lie algebra of meromorphic $\mathfrak{g}$-valued functions on Σ leaving invariant its sheaf of local sections by *endomorphism algebra* of Ψ, and denote it by $\mathrm{End}(\Psi)$.

In terms of flag configurations:

Let $\mathfrak{g} = Lie(G)$. Given a flag F consider the following filtration of $\mathfrak{g}$. Remind that V is a $\mathfrak{g}$-module. For every i consider a subspace $\tilde{\mathfrak{g}}_i \subseteq \mathfrak{g}$ such that $\tilde{\mathfrak{g}}_i F_j \subseteq F_{j+i}$ for every j. Then $\tilde{\mathfrak{g}}_i \subseteq \tilde{\mathfrak{g}}_{i+1}$ because $\tilde{\mathfrak{g}}_i F_j \subseteq F_{j+i} \subseteq F_{j+i+1}$, and $[\mathfrak{g}_i, \mathfrak{g}_k] \subseteq \mathfrak{g}_{i+k}$.

LEMMA: $\mathrm{End}(\Psi)$ is the subspace of the space of all $\mathfrak{g}$-valued meromorphic functions on Σ satisfying the following requirement for every $\gamma \in \Gamma$. Let L be such a function, and $L(z) = \sum L_i z^i$ be its Laurent expansion at a $\gamma \in \Gamma$. Then $L_i \in \tilde{\mathfrak{g}}_i, \ \forall i$.

Example: Lax operator algebras

Assume Ψ to be in diagonal form: $\Psi_\gamma = z^{h_\gamma}$, $\mathfrak{g} = \mathfrak{g}^\gamma_{-k} \oplus \ldots \oplus \mathfrak{g}^\gamma_k$ be the $\mathbb{Z}$-grading corresponding to h_γ, $\tilde{\mathfrak{g}}_j = \bigoplus\limits_{i \le j} \mathfrak{g}_i$ be the corresponding filtration: $\tilde{\mathfrak{g}}_{-k} \subset \ldots \subset \tilde{\mathfrak{g}}_{k-1} \subset \tilde{\mathfrak{g}}_k = \mathfrak{g}$.

Let L be a meromorphic $\mathfrak{g}$-valued function s.t. in a neighborhood of any $\gamma \in \Gamma$ $L(z) = \sum\limits_{i=1}^{\infty} L_i^\gamma z^i$, and

- $L_i^\gamma \in \tilde{\mathfrak{g}}_i^\gamma, \forall \gamma \in \Gamma$;
- L is holomorphic outside Γ and one more fixed finite set in Σ.

L is referred to as Lax operator algebra.

Matrix divisors and Hitchin systems

Consider filtrations $\ldots \subset \mathfrak{g}_i^\gamma \subset \mathfrak{g}_{i+1}^\gamma \subset \ldots$ as flags in $\mathfrak{g}$.

COROLLARY: Any $L \in \mathrm{End}(\Psi)$ is a section of the divisor $\mathrm{Ad}\,\Psi$.

Given a divisor $D = (\omega)$ where ω is a holomorphic 1-form on Σ, consider the sections $L \in \mathrm{End}(\Psi)$ holomorphic everywhere except at the divisor D. They are called Higgs fields.

Let L be a Higgs field, χ be an invariant polynomial on $\mathfrak{g}$. Then $\chi(L)$ is a meromorphic function on Σ holomorphic everywhere except at D. Scalar invariants of such functions (say, residues at the points in D, or coefficients of expansions over certain base) are called Hitchin Hamiltonians. This story has different continuations to Separation of Variables, Lax operator approach, Hamiltonian mechanics, and inverse scattering method.

Thank you!
Congratulations to Nikolai!