On (generalised) J-invariant of quadratic forms

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The plan of the talk

1. Definition(s) of J-invariant
   - Vishik's definition of J-invariant of quadratic forms
   - Generalised J-invariant
2. Morava J-invariant
   - Restrictions on J-invariant
   - Reduction to small ranks
   - Case of small ranks

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The variety of maximal totally isotropic subspaces

Let k be a field, $\operatorname{char} k \neq 2$, and let $q\colon V \to k$ be a nondegenerate quadratic form over k of dimension $2l+1$.

Let us denote $\mathrm{OGr}(l, q)$ the variety of totally isotropic subspaces of V of dimension l (this is a closed subvariety in the Grassmannian of V).

If q is split and P_l is the maximal parabolic subgroup of SO_{2l+1} corresponding to the fundamentalr root α_l (in the numbering of Bourbaki), then $\mathrm{OGr}(l, q) = \mathrm{SO}_{2l+1}/P_l$.

In general, $\mathrm{OGr}(l, q)$ is a twisted form of SO_{2l+1}/P_l.

Chow ring of $\mathrm{OGr}(l, q)$

Let us denote $\mathrm{Ch}^* = \mathrm{CH}^*(-; \mathbb{F}_2)$.

Theorem (Vishik)

$$\mathrm{Ch}^*(\mathrm{SO}_{2l+1}/P_l) \cong \mathbb{F}_2[e_1, e_2, \dots, e_l]/(e_i^2 = e_{2i} \mid 1 \le i \le l).$$

The answer does not depend on the base field k. Let L/k be a field extension splitting q and denote

$$\overline{\mathrm{Ch}}^*(\mathrm{OGr}(l, q)) = \mathrm{Im}\big(\mathrm{Ch}^*(\mathrm{OGr}(l, q)) \to \mathrm{Ch}^*(\mathrm{SO}_{2l+1}/P_l)\big).$$

Theorem (Vishik)

Consider the subring of $\mathrm{Ch}^*(\mathrm{SO}_{2l+1}/P_l)$ generated by $e_i \in \overline{\mathrm{Ch}}^*(\mathrm{OGr}(l, q))$. This subring coincides with $\overline{\mathrm{Ch}}^*(\mathrm{OGr}(l, q))$.

Definitions of J-invariant

Definition (Vishik)

Let $J(q)$ denote the subset of $\{1, \ldots, l\}$ such that $i \in J(q)$ iff $e_i \in \overline{\mathrm{Ch}}^*(\mathrm{OGr}(l, q))$.

$$\mathrm{Ch}^*(\mathrm{SO}_{2l+1}/P_l) = \mathbb{F}_2[e_1, e_3, \ldots, e_{2r-1}]/(e_{2i-1}^{2^{k_i}}),$$

where $r = \lfloor \frac{l+1}{2} \rfloor$, $k_i = \lfloor \log_2(\frac{2l}{2i-1}) \rfloor$.

Definition

Let j_i be the smallest integer such that $e_{2i-1}^{2^{j_i}} \in \overline{\mathrm{Ch}}^*(\mathrm{OGr}(l, q))$. Then

$$J'(q) = (j_1, \ldots, j_r).$$

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Hopf-theoretic interpretation of J-invariant

Let $\pi\colon \mathrm{SO}_{2l+1} \to \mathrm{SO}_{2l+1}/P_l$ denote the natural projection. Then

- $\pi^*\colon \mathrm{Ch}^*(\mathrm{SO}_{2l+1}/P_l) \to \mathrm{Ch}^*(\mathrm{SO}_{2l+1})$ is an isomorphism.
- Multiplication on SO_{2l+1} induces a co-multiplication on $\mathrm{Ch}^*(\mathrm{SO}_{2l+1})$, and e_i are primitive with respect to it.
- Let $\mathrm{Ch}^*(X) = \mathrm{Ch}^0(X) \oplus \widetilde{\mathrm{Ch}}^*(X)$.

Theorem (Petrov–Semenov)

The ideal generated by the image of $\widetilde{\mathrm{Ch}}^*(\mathrm{OGr}(l, q))$ in $\mathrm{Ch}^*(\mathrm{SO}_{2l+1})$ is a bi-ideal.

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Milnor–Moore Theorem

Theorem (Milnor–Moore)

If H is a non-negatively graded primitively generated commutative finite dimensional bi-algebra over $\mathbb{F}_p$, with $H^0 = \mathbb{F}_p$, then there is an isomorphism of bi-algebras of H with $\mathbb{F}_p[x_1, \ldots, x_s]/(x_i^{p^{e_i}})$ for some e_i, where x_i are primitive.

Corollary

Every bi-ideal in $\mathrm{Ch}^*(\mathrm{SO}_{2l+1})$ has form $(e_1^{2^{a_1}}, \ldots, e_{2r-1}^{2^{a_r}})$.

In particular, the ideal generated by the image of $\widetilde{\mathrm{Ch}}^*(\mathrm{OGr}(l, q))$ in $\mathrm{Ch}^*(\mathrm{SO}_{2l+1})$ has form $(e_1^{2^{j_1}}, \ldots, e_{2r-1}^{2^{j_r}})$ where $(j_1, \ldots, j_r) = J'(q)$.

Other cohomology theories

Let $\mathrm{char}\, k = 0$ and Ω^* denote algebraic cobordism of Levine–Morel. Consider $A^* = \Omega^* \otimes_{\Omega^*(\mathrm{Spec}\, k)} R$ for some $\Omega^*(\mathrm{Spec}\, k)$-algebra R.

Theorem (Petrov–Semenov)

Multiplication on SO_{2l+1} induces a co-multiplication on $A^*(\mathrm{SO}_{2l+1})$, and the ideal generated by the image of $\widetilde{A}^*(\mathrm{OGr}(l, q))$ in $A^*(\mathrm{SO}_{2l+1})$ is a bi-ideal.

Definition

We will denote $J'_A(q)$ the ideal generated by the image of $\widetilde{A}^*(\mathrm{OGr}(l, q))$ in $A^*(\mathrm{SO}_{2l+1})$ and call it the J-invariant of q with respect to the theory A.

Morava K-theories

$A^* = \mathrm{K}(n)^*$ (algebraic) Morava K-theory is another example of cohomology theory

Theorem (L.–Petrov–Sechin–Semenov)

$$\mathrm{K}(n)^*(\mathrm{SO}_{2l+1}) \cong \mathbb{F}_2[v_n^{\pm 1}][e_1, e_3, \ldots, e_{2r-1}]/(e_{2i-1}^{2^{k_i}}),$$

where $r = \min\left(2^{n-1}, \left\lfloor \frac{l+1}{2} \right\rfloor\right)$ *and*

$$k_i = \min\left(\left\lfloor \log_2\left(\frac{2^{n+1}-1}{2i-1}\right)\right\rfloor, \left\lfloor \log_2\left(\frac{2l}{2i-1}\right)\right\rfloor\right).$$

In particular, if $l \geq 2^n$ the natural map $\mathrm{SO}_{2^{n+1}-1} \to \mathrm{SO}_{2l+1}$ induces an isomorphism on $\mathrm{K}(n)$.

Bi-ideals in $\mathrm{K}(n)^*(\mathrm{SO}_{2l+1})$

Theorem

If $I \trianglelefteq \mathrm{K}(n)^*(\mathrm{SO}_{2l+1})$ is a bi-ideal, then $I = (e_1^{2^{a_1}}, \ldots, e_{2r-1}^{2^{a_r}})$ for some $0 \leq a_i \leq k_i$ and, moreover, I satisfies the condition

$$e_{2^n-1-2t} \in I \Rightarrow e_{2^n-1-t} \in I.$$

Definition

For $J'_{\mathrm{K}(n)}(q) = (e_1^{2^{j_1}}, \ldots, e_{2r-1}^{2^{j_r}}) \trianglelefteq \mathrm{K}(n)^*(\mathrm{SO}_{2l+1})$ denote $J'_{(n)}(q) = (j_1, \ldots, j_r)$.

Reduction to the case of small rank

For an $(2l+1)$-dimensional quadratic form q over k, $l \geq 2^n$, consider its generic splitting tower

$$k = k_0 \subset k_1 \subset \ldots \subset k_h$$

and $q_i = (q|_{k_i})_{\mathrm{an}}$, where $k_{i+1} = k_i(q_i)$. Assume that $\dim q_i \geq 2^{n+1} + 1$ for $i < i_0$ and $\dim q_{i_0} \leq 2^{n+1} - 1$. Denote $\widetilde{q} = q_{i_0} \perp \mathbb{H}^r$ such that $\dim \widetilde{q} = 2^{n+1} - 1$.

Theorem

Under the identification $\mathrm{K}(n)^*(\mathrm{SO}_{2^{n+1}-1}) \cong \mathrm{K}(n)^*(\mathrm{SO}_{2l+1})$ *the ideals* $J'_{\mathrm{K}(n)}(q)$ *and* $J'_{\mathrm{K}(n)}(\widetilde{q}\,)$ *coincide.*

Morava J-invariant for small ranks

Theorem

Let q denote $(2l+1)$-dimensional quadratic form, $l \leq 2^n - 1$. Then

$$J'_{(n)}(q) = J'(q).$$

In particular, $J'_{\mathrm{CH}}(q)$ satisfies the condition

$$e_{2^n-1-2t} \in J'_{\mathrm{CH}}(q) \Rightarrow e_{2^n-1-t} \in J'_{\mathrm{CH}}(q)$$

for $n \geq \log_2(l+1)$.

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for $n \geq \log_2(l+1)$.

Let S^i denote i-th Steenrod operation.
Let $k = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_t}$ where $a_1 = \nu_2(k)$ and $a_{i+1} = \nu_2(k - 2^{a_1} - \ldots - 2^{a_i})$, $0 \le a_1 < a_2 < \ldots < a_t$.

Theorem

In the notation above,

$$e_{2^n-1-k} = S^{2^{a_t}}(\ldots S^{2^{a_2}}(S^{2^{a_1}}(e_{2^n-1-2k}))\ldots) \in \mathrm{Ch}(\mathrm{SO}_m).$$

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