Quadratic pairs on Azumaya algebras over a scheme

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Algebraic groups, their friends and relations
in honour of Nikolai Vavilov, on occasion of his 70 birthday.

Санкт Петербург

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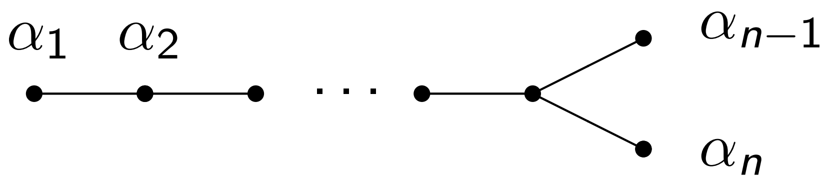
Outline

- (1) 2 is not always invertible ;
- (2) Algebraic groups in characteristic 2 ;



- (3) Reductive group schemes over a base where 2 is not invertible.

- It was initiated by Calmès and Fasel (2011).
- Example : the integral quadratic form Γ_8 is not diagonalizable.
- It arises as the norm form of the Coxeter octonions C and $\mathrm{Aut}(C)$ is the unique non-split $\mathbb{Z}$-group scheme of type G_2 (Springer-Veldkamp, 1960).
- Specific invariants are relevant.

Warm-up : quadratic étale algebras

That example shows well how the picture changes already for rings.

- Let R be a ring (commutative, unital). The quadratic étale R–algebras are classified by the group of étale cohomology ${H^1}_{\text{ét}}(R, \mathbb{Z}/2\mathbb{Z})$ or equivalently by the étale 2-covers of $\mathrm{Spec}(R)$.
- If 2 is invertible, $\mathbb{Z}/2\mathbb{Z}$ is isomorphic to $\boldsymbol{\mu}_2 \subset \mathbb{G}_m$. Kummer theory provides then the description of ${H^1}_{\text{ét}}(R, \boldsymbol{\mu}_2)$.
- This is the set of isomorphism classes of pairs $(\mathcal{L}, \theta)$ where $\mathcal{L}$ is an invertible R–module and $\theta : \mathcal{L} \otimes_R \mathcal{L} \xrightarrow{\sim} R$.
- If $\mathrm{Pic}(R) = 0$, we get $R^\times/(R^\times)^2$.
- In terms of algebras, this is $\mathcal{A} = R \oplus \mathcal{L}$ with multiplication law built from θ.

Quadratic étale algebras, II

- As the extreme opposite, if R is a $\mathbb{F}_2$–algebra, Artin-Schreier theory shows that $H^1_{\text{ét}}(R, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} R/\mathcal{P}(R)$ where $\mathcal{P}(x) = x + x^2$.
- In this case, the Picard group $\mathrm{Pic}(R)$ is not involved.
- In terms of algebras, we associate to $a \in R$ the quadratic R–algebra $S_a = R[t]/(t^2 + t + a)$.
- The more advanced case is when 2 is non zero and non invertible. It has been investigated by Small, by Waterhouse and others using cohomological techniques. It is quite complicated since it covers the two above cases.

Quadratic pairs, the field case

The quadratic pairs is one of the main advance of the book of involutions (Knus, Merkurjev, Rost, Tignol, 1998).

- Let A be a central simple k–algebra where k is a field.
- A *quadratic pair* on A is a pair (σ, f) where σ is an involution of the first kind and f is a linear function $f : \mathrm{Sym}(A, \sigma) \to k$ on the symmetric elements of A (those $a \in A$ satisfying $\sigma(a) = a$), such that
- $$f(a + \sigma(a)) = \mathrm{Trd}_A(a) \quad \forall a \in A.$$
- We have $2f = \mathrm{Trd}_A$ so f is unique if 2 is invertible.

Quadratic pairs, the field case, II

We fix the terminology for involutions following Calmès and Fasel. We continue to deal with a c.s.a. A with an involution σ of first kind.

- Let K/k be a splitting field, for A, that is, there exists an isomorphism $A \otimes_k K \cong \mathrm{M}_n(K)$. The involution $\sigma \otimes 1$ on $\mathrm{M}_n(K)$ is then the adjoint involution of some bilinear form b.
- σ is *orthogonal* if b is symmetric,
- σ is *weakly symplectic* if b is skew-symmetric,
- σ is *symplectic* if b is alternating, i.e., $b(x, x) = 0$ for all x.
- In characteristic $\neq 2$, weakly symplectic is the same as symplectic and this agrees with the classical conventions.
- In characteristic 2, orthogonal is the same as weakly symplectic. Symplectic is the same notion as in the book of involutions but with the new conventions a symplectic involution is orthogonal !
- Of course it does not depend of the various choices.

Field case, III

From now on we deal mostly with the case of an orthogonal involution σ on A of even degree (for simplicity).

- Let (V, q) be a even dimensional regular quadratic form. We can attach to it the bilinear form b_q, an orthogonal involution σ_q and an isomorphism $\varphi_q : V \otimes_k V \xrightarrow{\sim} \mathrm{End}_k(V)$ defined by $\varphi_q(v \otimes w).x = v\, b_q(w, x)$.
- There exists a unique linear form f_q satisfying $f_q(\varphi_q(v \otimes v)) = q(v) \qquad \forall v \in V.$
- Then (σ_q, f_q) is a quadratic pair on $\mathrm{End}_k(V)$ and $O(q) = O(\sigma_q, f_q)$.
- If (σ, f) is an orthogonal quadratic pair on $\mathrm{End}_k(V)$, then $(\sigma, f) = (\sigma_q, f_q)$ for a unique q up to scaling.

Field case, IV

We continue our reading of the book of involutions.

- If A is a c.s.a. of even degree $2n \geq 4$ equipped with an orthogonal quadratic pair (σ, f), its orthogonal group $\mathrm{O}(A, \sigma, f) \subset \mathrm{GL}_1(A)$ is smooth and its neutral component $\mathrm{O}(A, \sigma, f)^+$ is semisimple of type D_n.
- Conversely all classical (that is non trialitarian) forms of SO_{2n} are of this shape and there is unicity.
- There was a previous description of groups of type D_n in characteristic 2 by generalized quadratic forms (Seip-Hornix, Tits). One advantage of quadratic pairs is for example when dealing with base change.

Over a scheme

Following Calmès-Fasel, the results above do generalize to the scheme setting. Let S be a base scheme. Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra of even degree equipped with an orthogonal involution σ.

- Orthogonal means that there is a flat cover $S' \to S$ such that $\mathcal{A}_{S'} \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_{S'}}(\mathcal{E}')$ and that $\sigma_{S'}$ is adjoint to a symmetric bilinear form on the locally free sheaf of finite rank $\mathcal{E}'$.
- A *quadratic pair* on $\mathcal{A}$ is a pair (σ, f) where σ is an orthogonal involution of the first kind and f is a linear function $f \colon \mathcal{S}ym(\mathcal{A}, \sigma) \to \mathcal{O}_S$ such that
- for each affine S–scheme T

$$f(a + \sigma(a)) = \mathrm{Trd}_{\mathcal{A}(T)}(a) \quad \text{for all } a \in \mathcal{A}(T).$$

- Attached to a regular quadratic form $(\mathcal{M}, q)$ of even rank, we can attach a quadratic pair (σ_q, f_q) on $\mathrm{End}_{\mathcal{O}_S}(\mathcal{M})$ exactly as in the field case.

Over a scheme,II

We continue our reading of Calmès-Fasel's paper.

- If $\mathcal{A}$ is of constant rank $2n \geq 4$ and (σ, f) is a (orthogonal) quadratic pair on $\mathcal{A}$, then $\mathrm{O}(\mathcal{A}, \sigma, f)$ is a smooth affine S–group scheme and its neutral component $\mathrm{O}(\mathcal{A}, \sigma, f)^+$ is semisimple of type D_n.
- Conversely if $\mathfrak{G}$ is a classical S–form of the Chevalley group SO_{2n}, it is of the above shape in an unique manner.

Description of quadratic pairs for a fixed involution

Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra of even rank equipped with an orthogonal involution σ.

- Question I : can we extend σ to a quadratic pair (σ, f) ?
- Question II : if it extends, can we describe all possible f ?
- The second question is easier and requires already some investigation.
- It is convenient to work in the flat topology setting.

Various sheaves

We continue with $(\mathcal{A}, \sigma)$ over S and work with fppf sheaves over S.

- We associate the following $\mathcal{O}$–modules :

$$\begin{aligned} \mathcal{S}ym_{\mathcal{A},\sigma} &= \mathrm{Ker}(\mathrm{Id} - \sigma) \quad (\textit{symmetric elements}), \\ \mathcal{A}\ell t_{\mathcal{A},\sigma} &= \mathrm{Im}(\mathrm{Id} - \sigma) \quad (\textit{alternating elements}), \\ \mathcal{S}kew_{\mathcal{A},\sigma} &= \mathrm{Ker}(\mathrm{Id} + \sigma) \quad (\textit{skew-symmetric elements}), \\ \mathcal{S}ymd_{\mathcal{A},\sigma} &= \mathrm{Im}(\mathrm{Id} + \sigma) \quad (\textit{symmetrized elements}), \end{aligned}$$

  where Im(__) is the image fppf-sheaf.
- Using the trace form $\mathcal{S}ym_{\mathcal{A},\sigma}$, $\mathcal{A}\ell t_{\mathcal{A},\sigma}$ are locally direct summands of $\mathcal{A}$. Furthermore we have $\mathcal{S}ym^{\perp}_{\mathcal{A},\sigma} = \mathcal{A}\ell t_{\mathcal{A},\sigma}$ and $\mathcal{A}\ell t^{\perp}_{\mathcal{A},\sigma} = \mathcal{S}ym_{\mathcal{A},\sigma}$ where the $\perp$ is related to the symmetric regular bilinear form $(x, y) \mapsto \mathrm{Trd}_{\mathcal{A}}(xy)$.
- However $\mathcal{S}kew_{\mathcal{A},\sigma}$ and $\mathcal{S}ymd_{\mathcal{A},\sigma}$ are not in general locally direct summands of $\mathcal{A}$ neither finite locally free $\mathcal{O}$–modules.

Other pairs

We continue with $(\mathcal{A}, \sigma)$ over S.

- If $f, f' : \mathcal{S}ymd_{\mathcal{A},\sigma} \to \mathcal{O}_S$ complete both σ, we have $(f' - f)(a + \sigma(a)) = 0$ for each affine S–scheme T and each $a \in \mathcal{A}(T)$.
- Equivalently $f' - f$ is zero on the $\mathcal{O}$-subsheaf $\mathcal{S}ymd_{\mathcal{A},\sigma}$ of $\mathcal{S}ym_{\mathcal{A},\sigma}$.
- In conclusion the f's completing σ is principal homogeneous under $H^0\big(S, (\mathcal{S}ym_{\mathcal{A},\sigma}/\mathcal{S}ymd_{\mathcal{A},\sigma})^\vee\big)$.
- Since $\mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma} \cong (\mathcal{S}ym_{\mathcal{A},\sigma}/\mathcal{S}ymd_{\mathcal{A},\sigma})^\vee$, this is equivalently principal homogeneous under $H^0\big(S, \mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma}\big)$.

Existence problem

- Assuming we can complete σ locally for the fppf topology (or equivalently for the Zariski topology), there is then a fppf sheaf torsor $\mathcal{T}$ under the sheaf $\mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma}$ which encodes whether σ is extendable to a quadratic pair.
- There is then a local study and we can wonder to identify a class $\omega(\mathcal{A},\sigma) \in H^1_{\mathrm{fppf}}\big(S, \mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma}\big)$ associated to the torsor $\mathcal{T}$.
- We start with the affine case, that is with $S = \mathrm{Spec}(R)$ and the Azumaya R–algebra $A = \mathcal{A}(S)$.
- **Lemma**. The following are equivalent :

  (a) σ extends to a quadratic pair ;

  (b) $1_A = a + \sigma(a)$ for some $a \in A$;

  (c) There exists a fppf cover R' of R such that $A_{R'} \cong \mathrm{End}_{R'}(M')$ and $\sigma_{R'}$ is adjoint to a regular quadratic form q' on M'.

Existence problem, II

- **Lemma**. The following are equivalent :
  (a) σ extends to a quadratic pair ;
  (b) $1_A = a + \sigma(a)$ for some $a \in A$;
  (c) There exists a fppf cover R' of R such that $A_{R'} = \mathrm{End}_{R'}(M')$ and $\sigma_{R'}$ is adjoint to a regular quadratic form q' on M'.
- Let us prove the equivalence $(a) \iff (b)$. If $1_A = a + \sigma(a)$ for some $a \in A$, then $f_a(x) = \mathrm{Trd}_A(ax)$ completes σ.
- Conversely if σ extends to a quadratic pair (σ, f), f is the restriction of a linear form $\widetilde{f} : A \to R$. Then there exists $a \in \mathcal{A}$ such that $\widetilde{f}(x) = \mathrm{Trd}_A(ax)$.
- We have $\widetilde{f}(\sigma(x)) = \mathrm{Trd}_A(\sigma(a)x)$ so that $\mathrm{Trd}_A(x) = f(x + \sigma(x)) = \mathrm{Trd}_A((a + \sigma(a))x)$ for all $x \in A$. Thus $1 = a + \sigma(a)$ as desired.
- This condition is not always satisfied starting from the case of $\mathbb{F}_2$ and rank 4. More precisely there are orthogonal involutions which are not symplectic.

Cohomological obstructions

- We assume that $(\mathcal{A}, \sigma)$ is *locally quadratic*, that is, there exists a Zariski (or flat) cover of S by affine schemes $(T_i)_{i\in I}$ such that $(\mathcal{A}(T_i), \sigma)$ satisfies the equivalent condition of the above statement.
- This condition is local for the flat topology.
- We have by construction an exact sequence

$$0 \longrightarrow \mathcal{S}kew_{\mathcal{A},\sigma} \longrightarrow \mathcal{A} \xrightarrow{1+\sigma} \mathcal{S}ymd_{\mathcal{A},\sigma} \longrightarrow 0$$

- The idea is simple. We consider the long exact sequence

$$0 \longrightarrow H^0(S, \mathcal{S}kew_{\mathcal{A},\sigma}) \longrightarrow H^0(S, \mathcal{A}) \xrightarrow{1+\sigma} H^0(S, \mathcal{S}ymd_{\mathcal{A},\sigma})$$

$$\xrightarrow{\delta} H^1_{\mathrm{fppf}}(S, \mathcal{S}kew_{\mathcal{A},\sigma}).$$

- We have $1_{\mathcal{A}} \in H^0(S, \mathcal{S}ymd_{\mathcal{A},\sigma})$ and put $\Omega(\mathcal{A}, \sigma) = \delta(1_{\mathcal{A}}) \in H^1_{\mathrm{fppf}}(S, \mathcal{S}kew_{\mathcal{A},\sigma})$.

Cohomological obstructions, II

$$\begin{array}{ccccccccc} 0 \longrightarrow H^0(S, \mathcal{S}kew_{\mathcal{A},\sigma}) \longrightarrow H^0(S,\mathcal{A}) \xrightarrow{1+\sigma} H^0(S, \mathcal{S}ymd_{\mathcal{A},\sigma}) \xrightarrow{\delta} H^1_{\mathrm{fppf}}(S, \mathcal{S}kew_{\mathcal{A},\sigma}) \\ \downarrow \\ H^1_{\mathrm{fppf}}(S, \mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}\ell t_{\mathcal{A},\sigma}). \end{array}$$

- $\Omega(\mathcal{A},\sigma) = \delta(1_{\mathcal{A}}) \in H^1_{\mathrm{fppf}}(S, \mathcal{S}kew_{\mathcal{A},\sigma})$ and denote by $\omega(A,\sigma)$ its image in $H^1_{\mathrm{fppf}}(S, \mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}\ell t_{\mathcal{A},\sigma})$.
- Both invariants are obviously killed by 2 and are zero if 2 is invertible on S.
- **Theorem.** (a) The following are equivalent :

  (i) σ extends to a quadratic pair ;

  (ii) $\omega(\mathcal{A},\sigma) = 0$.

  (b) The following are equivalent :

  (iii) σ extends to a quadratic pair (σ, f) such that f extends to $\mathcal{A}$;

  (iv) $\Omega(\mathcal{A},\sigma) = 0$.

Examples

It remains to explain why the above obstructions are a non-trivial construction.

- More precisely we are interested in the construction of an example where $\omega(\mathcal{A}, \sigma) \neq 0$ and and also of an example where $\omega(\mathcal{A}, \sigma) = 0$ but $\Omega(\mathcal{A}, \sigma) \neq 0$.
- In both cases we shall deal with a certain projective variety over an algebraically closed field k of characteristic 2 and work directly with some $(\mathcal{A}, \sigma)$.
- We take an embedding $\mathbb{F}_4 \subset k$; it induces an embedding $\Gamma := \mathrm{PGL}_2(\mathbb{F}_4) \subset G = \mathrm{PGL}_2$.
- On the other hand, Serre's theorem provides a Galois cover $Y \to X$ of group Γ where Y, X are smooth connected projective k–varieties (of dimension one if we want so).
- We consider the PGL_2–torsor $P = Y \times^{\Gamma} \mathrm{PGL}_2$ over X and denotes by $\mathcal{A}$ the associated quaternion Azumaya $\mathcal{O}_X$–algebra. It comes with the canonical symplectic involution σ which is also orthogonal.

Examples, II

$\mathbb{F}_4 \subset k$ and $\Gamma := \mathrm{PGL}_2(\mathbb{F}_4) \subset G = \mathrm{PGL}_2$.

- $Y \to X$ is a Galois cover of group Γ where Y, X are smooth connected projective varieties.
- We consider the PGL_2–torsor $P = Y \times^{\Gamma} \mathrm{PGL}_2$ over X and denotes by $\mathcal{A}$ the associated quaternion Azumaya $\mathcal{O}_X$–algebra coming with its canonical (orthogonal) involution σ.
- **Claim.** $(\mathcal{A}, \sigma)$ is locally quadratic but cannot be extended to a quadratic pair, so that $\omega(A, \sigma) \neq 0$.
- *Sketch of proof.* $(\mathcal{A}, \sigma)$ is locally quadratic since it is a descent of $(\mathrm{M}_2(\mathcal{O}_Y), \mathrm{tr} + id)$ and $\mathrm{tr} + id$ arises from the hyperbolic form on $\mathcal{O}_Y^2$.
- Assume that $(\mathcal{A}, \sigma)$ is extendable to $(\mathcal{A}, \sigma, f)$ over X. By base change we get then $(\mathcal{A}, \sigma, f)_Y \cong (\mathrm{M}_2(\mathcal{O}_Y), \mathrm{tr} + id, \widetilde{f})$ where $\widetilde{f}$ is Γ-equivariant (for the adjoint action).

End of proof, II

$\mathbb{F}_4 \subset k$ and $\Gamma := \mathrm{PGL}_2(\mathbb{F}_4) \subset G = \mathrm{PGL}_2$.

- *Sketch of proof.* Assume that $(\mathcal{A}, \sigma)$ is extendable to $(\mathcal{A}, \sigma, f)$ over X. By base change we get then $(\mathcal{A}, \sigma, f)_Y \cong (\mathrm{M}_2(\mathcal{O}_Y), \mathrm{tr} + id, \widetilde{f})$ where $\widetilde{f}$ is Γ-equivariant (for the adjoint action).
- We have $O_Y^3 = \begin{pmatrix} a & b \\ c & a \end{pmatrix} = \mathcal{S}ym_{\mathrm{M}_2(\mathcal{O}_Y), \mathrm{tr} + id}$.
- Since $H^0(Y, O_Y) = k$, $\widetilde{f}$ is determined by the global sections, that is, a map $f_0 : k^3 = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \to k$ which is furthermore Γ-equivariant.
- By elementary computations, f_0 must be zero.
- Thus $\widetilde{f} = 0$, contradiction.

- Summarizing we found a non-trivial cohomological obstruction for extending $(\mathcal{A}, \sigma)$ to $(\mathcal{A}, \sigma, f)$.

- Спасибо !

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