Regular bi-interpretability of Chevalley groups

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Algebraic groups, their friends and relations
(In honour of Nikolay Vavilov
on occasion of his 70 birthday)
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Elementary equivalence

Definition

Two models $\mathcal{U}_1$ and $\mathcal{U}_2$ of the same first order language are called *elementary equivalent* (notation: $\mathcal{U}_1 \equiv \mathcal{U}_2$), if for every first order sentence φ of this language φ holds in $\mathcal{U}_1$ if and only if it holds in $\mathcal{U}_2$.

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Example (O. Kharlampovich, A. Myasnikov; Z. Sela, 2006)

All non-abelian free groups are elementary equivalent.

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Theorem (A.I. Maltsev, 1961)

Two groups $\mathcal{G}_n(R_1)$ *and* $\mathcal{G}_m(R_2)$ *(where* $\mathcal{G} = \mathrm{GL}, \mathrm{SL}, \mathrm{PGL}, \mathrm{PSL}$*,* $n, m \geqslant 3$*,* R_1, R_2 *are fields of characteristics* 0*) are elementarily equivalent if and only if* $m = n$ *and the fields* R_1 *and* R_2 *are elementary equivalent.*

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Proof methods:

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- Show directly that the group $\mathcal{G}_n(R)$ is absolutely interpretable in R (easy part);
- Find formulas which differ $\mathcal{G}_n$ and $\mathcal{G}_m$ for $n \neq m$;
- Show directly that R is regularly interpretable in $\mathcal{G}_n(R)$ (requires to consider many cases).

Isomorphism theorem applying to Maltsev-type theorems proofs

Theorem (Keisler–Shelah Isomorphism Theorem, 1971–1974)

Two models $\mathcal{U}_1$ and $\mathcal{U}_2$ are elementary equivalent if and only if there exists an ultrafilter $\mathcal{F}$ such that

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- $\mathcal{G}_n(\prod_{\mathcal{F}} R_1) \cong \mathcal{G}_m(\prod_{\mathcal{F}} R_2) \Longrightarrow n = m$ and $\prod_{\mathcal{F}} R_1 \cong \prod_{\mathcal{F}} R_2 \Longrightarrow n = m$ and $R_1 \equiv R_2$.

Several generalizations of Maltsev theorem

- **K.I. Beidar, A.V. Mikhalev, 1992:**
  The groups $\mathrm{GL}_n(R_1)$ and $\mathrm{GL}_m(R_2)$ ($n, m \geqslant 3$, R_1, R_2 are skewfields or $n, m \geqslant 2$, R_1, R_2 are infinite fields) are elementary equivalent if and only if
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- **V.A. Bragin, E.I. Bunina, 2013:**
  The groups $\mathrm{GL}_n(R_1)$ and $\mathrm{GL}_m(R_1)$ (R_1, R_2 are rings with finite number of central idempotents and with 1, $n, m \geqslant 4$ or with $1/2$, $n, m \geqslant 3$) are elementary equivalent if and only if there exists central idempotents $e \in R_1$ and $f \in R_2$ such that
  $eM_n(R_1) \equiv fM_m(R_2)$ and $(1-e)M_n(R_1) \equiv (1-f)M_m(R_2)^{op}$.

Chevalley groups. Definitions

All Chevalley groups are linear groups over commutative rings.
Every Chevalley group $G_\pi(R, \Phi)$ is constructed by:
— a semisimple complex Lie algebra $\mathcal{L}$ with the root system Φ;
— a linear representation $\pi : \mathcal{L} \to \mathrm{GL}_N(\mathbb{C})$;
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Example

A_l — $\mathrm{SL}_{l+1}(R)$, $\mathrm{PGL}_{l+1}(R)$, $\ldots$;

B_l — $\mathrm{Spin}_{2l+1}(R)$, $\mathrm{SO}_{2n+1}(R)$;

C_l — $\mathrm{Sp}_{2l}(R)$, $\mathrm{PSp}_{2l}(R)$;

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Elementary Chevalley group $E_\pi(\Phi, R)$ is the subgroup of $G_\pi(\Phi, R)$ generated by special elements (*root unipotents*) $x_\alpha(t)$, $\alpha \in \Phi$, $t \in R$.

Maltsev-type theorems for Chevalley groups

- **E.I. Bunina, 2004:**
  If two Chevalley groups are constructed by irreducible root systems Φ_1, Φ_2, weight lattices Λ_1, Λ_2 and infinite fields K_1, K_2 of characteristics $\neq 2$, then

$$G_{\pi_1}(\Phi_1, K_1) \equiv G_{\pi_2}(\Phi_2, K_2) \quad \Longleftrightarrow \quad \begin{cases} \Phi_1 = \Phi_2, \\ \Lambda_1 = \Lambda_2, \\ K_1 \equiv K_2. \end{cases}$$

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- **E.I. Bunina, 2009:**
  If two Chevalley groups are constructed by irreducible root systems Φ_1, Φ_2 of ranks > 1, weight lattices Λ_1, Λ_2 and local commutative rings R_1, R_2 with $1/2$ and for the root system G_2 with $1/3$, then

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- **E.I. Bunina, 2018:**
  Let $G_{\mathrm{ad}}(\Phi_1, R_1)$ and $G_{\mathrm{ad}}(\Phi_2, R_2)$ be two (elementary) adjoint Chevalley groups with undecomposable root systems Φ_1, Φ_2 of ranks > 1, R_1, R_2 be commutative rings with 1.
  Suppose that for the root systems A_2, B_l, C_l or F_4 the ring contains $1/2$, for G_2 it contains $1/2$ and $1/3$.
  Then

$$G_{\mathrm{ad}}(\Phi_1, R_1) \equiv G_{\mathrm{ad}}(\Phi_2, R_2) \quad \Longleftrightarrow \quad \begin{cases} \Phi_1 = \Phi_2, \\ R_1 \equiv R_2. \end{cases}$$

Elementary definability

Definition

Some class $\mathcal{G}$ of algebraic systems of the same language $\mathcal{L}$ is called *elementary* definable if for any $G \in \mathcal{G}$ and for any algebraic system H of the language $\mathcal{L}$ if $H \equiv G$, then $H \in \mathcal{G}$.

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(1) The class of all Abelian groups is elementary definable.
(2) More generally **any variety of groups is elementary definable**: if $\mathcal{G}$ is some variety of groups defined by a system of identity relations Λ and a group H is elementarily equivalent to some $G \in \mathcal{G}$, then H satisfies the same system Λ, therefore, $H \in \mathcal{G}$.

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On the other hand for general linear groups even over fields there is no elementary definability: there exists a group $H \equiv \mathrm{GL}_n(R)$ which is not GL itself.

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We will discuss it later.

First-order ridigity

A new round of development of this topic has appeared recently in the papers of

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We know that classification up to elementary equivalence is always strictly wider than classification up to isomorphism. But taking finitely generated structures (groups or rings or anything else) there exists a possibility that for some of them elementary equivalence could imply isomorphism.

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Definition (N. Avni, A. Lubotzky, C. Meiri, 2019)

A finitely generated group (ring or other structure) A is *first-order rigid* if any other finitely generated group (ring or other structure) elementarily equivalent to A is isomorphic to A.

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Theorem (A. Myasnikov and M. Sohrabi, 2022)

Let $\mathcal{O}$ be the ring of integers of a number field $\mathbb{F}$ of finite degree, and $n \geqslant 3$.
Then the groups $\mathrm{SL}_n(\mathbb{Z})$, $\mathrm{SL}_n(\mathbb{Q})$, $\mathrm{SL}_n(\mathbb{F})$, $\mathrm{SL}_n(\mathcal{O})$ are first order rigid.

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An infinite finitely generated structure $\mathcal{U}$ is *quasi finitely axiomatizable* (*QFA*) if there is a first order sentence φ such that
(1) $\mathcal{U} \models \varphi$;
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Example (examples of QFA groups)

(1) nilpotent groups $UT_3(\mathbb{Z})$;
(2) metabeilan groups $\langle a, d \mid d^{-1}ad = a^m \text{ for any } m \geqslant 3 \rangle$;
(3) permutation groups: the subgroup of the group of permutations of $\mathbb{Z}$ generated by the successor function and the transposition $(0, 1)$.

Interpretability

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A model $\mathcal{B}$ is **interpretable in a model** $\mathcal{A}$ if the elements of $\mathcal{B}$ can be represented by tuples in a definable relation $\mathcal{D}$ on $\mathcal{A}$, in such a way that equality of $\mathcal{B}$ becomes an $\mathcal{A}$-definable equivalence relation $\mathcal{E}$ on $\mathcal{D}$, and the other atomic relations on $\mathcal{B}$ are also definable.

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Example

A simple example is the difference group construction: for instance, $(\mathbb{Z}, +)$ can be interpreted in $(\mathbb{N}, +)$, where the relation $\mathcal{D}$ is $\mathbb{N} \times \mathbb{N}$, addition is component-wise and $\mathcal{E}$ is the relation given by $(n, m)\mathcal{E}(n', m') \iff n' + m = n + m'$.

Bi-interpretability

Suppose structures $\mathcal{A}, \mathcal{B}$ in finite signatures are given, as well as interpretations of $\mathcal{A}$ in $\mathcal{B}$, and vice versa. Then an isomorphic copy $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ can be defined in $\mathcal{A}$, by "decoding" $\mathcal{A}$ from the copy of $\mathcal{B}$ defined in $\mathcal{A}$.

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Similarly, an isomorphic copy $\widetilde{\mathcal{B}}$ of $\mathcal{B}$ can be defined in $\mathcal{B}$. An isomorphism $\Phi : \mathcal{A} \cong \widetilde{\mathcal{A}}$ can be viewed as a relation on $\mathcal{A}$, and similarly for an isomorphism $\mathcal{B} \cong \widetilde{\mathcal{B}}$.

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Definition (bi-interpretability)

We say that $\mathcal{A}$ and $\mathcal{B}$ are **bi-interpretable** (with parameters) if there are such isomorphisms that are first-order definable.

Regular interpretability

Definition

We say that a structure $\mathcal{A}$ is interpreted in a given structure $\mathcal{B}$ **uniformly** with respect to a subset $D \subseteq \mathcal{B}^k$ if there is one interpretation of $\mathcal{A}$ in $\mathcal{B}$ with any tuple of parameters $\overline{p} \in D$.

If $\mathcal{A}$ is interpreted in $\mathcal{B}$ uniformly with respect to a $\emptyset$-definable subset $D \subseteq \mathcal{B}^k$ then we say that $\mathcal{A}$ is **regularly interpretable** in $\mathcal{B}$ and write in this case $\mathcal{A} \cong \Gamma(\mathcal{B}, \varphi)$, provided D is defined by φ in $\mathcal{B}$.

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An important application of regular interpretability is the following:

Theorem

If $\mathcal{A}_1 = \Gamma(\mathcal{B}_1, \varphi)$ and $\mathcal{A}_2 = \Gamma(\mathcal{B}_2, \varphi)$ are two regular interpretations with the same interpretation and the same formula φ, then $\mathcal{B}_1 \equiv \mathcal{B}_2$ implies $\mathcal{A}_1 \equiv \mathcal{A}_2$.

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(1) $\mathcal{A}$ and $\mathcal{B}$ are regularly interpretable in each other, so $\mathcal{A} \cong \Gamma(\mathcal{B}, \varphi)$ and $\mathcal{B} \cong \Delta(\mathcal{A}, \psi)$. By transitivity $\mathcal{A}$, as well as $\mathcal{B}$, is regularly interpretable in itself, so we can write

$$\mathcal{A} \cong (\Gamma \circ \Delta)(\mathcal{A}, \varphi^*) \text{ and } \mathcal{B} \cong (\Delta \circ \Gamma)(\mathcal{B}, \psi^*).$$

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(2) There is a formula $\theta(\overline{y}, x, \overline{z})$ in the language of $\mathcal{A}$ such that for every tuple p^* satisfying $\varphi^*(\overline{z})$ in $\mathcal{A}$ the formula $\theta(\overline{y}, x, p^*)$ defines in $\mathcal{A}$ the isomorphism $\overline{\mu}_{\Gamma \circ \Delta} : (\Gamma \circ \Delta)(\mathcal{A}, p^*) \to \mathcal{A}$ and there is a formula $\sigma(\overline{u}, x, \overline{\ })$ in the language of $\mathcal{B}$ such that for every tuple q^* satisfying $\psi^*(\overline{v})$ in $\mathcal{B}$ the formula $\sigma(\overline{v}, x, q^*)$ defines in $\mathcal{B}$ the isomorphism $\overline{\mu}_{\Delta \circ \Gamma} : (\Delta \circ \Gamma)(\mathcal{B}, q^*) \to \mathcal{B}$.

Regular bi-interpretability and elementary definability

If linear groups (or another derivative structures) of any concrete types over some classes of fields/rings are regularly bi-interpretable with the corresponding rings, then this class of groups (structures) is elementary definable:

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Theorem

If for some class of rings $\mathcal{R}$, closed under elementary equivalence, and the class $\mathcal{G} = \{G(R) \mid R \in \mathcal{R}\}$ (where $G(R)$ is any type of derivative groups: linear groups, Chevalley groups, automorphism groups, etc.) all the groups $G(R)$ are regularly interpretable with the corresponding rings R with the same interpretations, then the class $\mathcal{G}$ is elementarily definable in class of all groups, i. e., for any group H such that $H \equiv G(R)$ for some $R \in \mathcal{R}$ there exists a ring $R' \equiv R$ such that $H \cong G(R')$.

Bi-interpretability with the ring $\mathbb{Z}$ and QFA property

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Recall that by a *number field* we mean a finite extension of $\mathbb{Q}$. By the *ring of integers* $\mathcal{O}$ of a number field F we mean the subring of F consisting of all roots of monic polynomials with integer coefficients.

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By a known result any ring of integers $\mathcal{O}$ of a number field F is bi-interpretable with $\mathbb{Z}$ (and therefore is QFA).

Regular bi-interpretability of linear and triangular groups over rings of integers

A. Myasnikov and M. Sohrabi proved that the groups $\mathrm{SL}_n(\mathcal{O})$, $n \geqslant 3$ are bi-interpretable (with parameters) with the ring $\mathcal{O}$ and therefore with $\mathbb{Z}$; the groups $\mathrm{GL}_n(\mathcal{O})$ and $T_n(\mathcal{O})$, $n \geqslant 3$, are bi-interpretable with $\mathcal{O}$ (and with $\mathbb{Z}$) only if $\mathcal{O}^*$ is finite. Consequently in all good cases these linear groups are QFA.

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Moreover they proved that for all these "good cases" the problem of general elementary definability has positive solution:

Theorem (A. Myasnikov, M. Sohrabi, 2021)

If $n \geqslant 3$, $\mathcal{O}$ is the ring of integers of some number field F, H is an arbitrary groups, $H \equiv \mathrm{SL}_n(\mathcal{O})$ or $\mathcal{O}^*$ is finite and $H \equiv \mathrm{GL}_n(\mathcal{O})$ or $H \equiv T_n(\mathcal{O})$. Then $H \cong \mathrm{SL}_n(R)$ (or $\mathrm{GL}_n(R)$, $T_n(R)$ respectively) for some ring R such that $R \equiv \mathcal{O}$.

Really they proved in their work that these groups are **regularly** bi-interpretable with the corresponding rings.

Bi-interpretability of Chevalley groups over integral domains

Theorem (D. Segal. K. Tent, 2020)

Let $G(\cdot)$ be a simple Chevalley-Demazure group scheme of rank at least two, and let R be an integral domain. Then R and $G(R)$ are bi-interpretable provided either
(1) G is adjoint, or (2) $G(R)$ has finite elementary width, assuming in case G is of type E_6, E_7, E_8, or F_4 that R has at least two units.

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Corollary

Assume that G and R satisfy the hypotheses of the previous theorem. If R is first order rigid (resp., QFA), in
(1) the class o finitely generated rings,
(2) the class of profinite rings,
(3) the class of locally compact topological rings,
then $G(R)$ has the analogous property in (1) the class of finitely generated groups, (2) the class of profinite groups, (3) the class of locally compact topological groups.

Regular bi-interpretability of Chevalley groups over local rings

Theorem (Bunina, 2022)

If $G(R) = G_\pi(\Phi, R)$ $(E(R) = E_\pi(\Phi, R))$ is an (elementary) Chevalley group of rank > 1, R is a local ring (with $\frac{1}{2}$ for the root systems $\mathrm{B}_l, \mathrm{C}_l, \mathrm{F}_4, \mathrm{G}_2$ and with $\frac{1}{3}$ for G_2), then the group $G(R)$ (or $(E(R))$ is regularly bi-interpretable with R.

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Corollary (elementary definability of Chevalley groups)

The class of Chevalley groups over local rings is elementarily definable, i. e., if $G(R) = G_\pi(\Phi, R)$ is a Chevalley group of rank > 1, R over a local ring R (with $\frac{1}{2}$ for the root systems $\mathrm{A}_2, \mathrm{B}_l, \mathrm{C}_l, \mathrm{F}_4, \mathrm{G}_2$ and with $\frac{1}{3}$ for G_2) and for an arbitrary group H we have $H \equiv G(R)$, then $H \equiv G_\pi(\Phi, R')$ for some local ring R', which is elementarily equivalent to R.

Sketch of the proof. Step 1

1.1. An elementary Chevalley group over a local ring is the commutant of the Chevalley group $G = G_\pi(\Phi, R)$ of a bounded length, therefore an elementary subgroup is absolutely interpretable in G.

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$\Longrightarrow$

In the next steps **we can work in the group E_{ad}.**

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Remark

This step is possible for all Chevalley groups where the commutant has a bounded length.

Sketch of the proof. Step 2

Prove that **all elementary unipotent subgroups** $X_\alpha = \{x_\alpha(t) \mid t \in R\}$ **are definable in** $E_{\text{ad}} = E_{\text{ad}}(\Phi, R)$ with parameters $\overline{x} = \{x_\alpha(1) \mid \alpha \in \Phi\}$.

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Remark

This step can be possible for all Chevalley groups over all commutative rings.

Gauss decomposition and first order formulas

Proposition

(1) Every element x of G over a local ring R can be represented as

$$x = utvu', \text{ where } u, u' \in U(R),\ v \in V(R),\ t \in T(R);$$

(2) There exists first order formulas $\varphi(\dots)$ of $6n+2l$ arguments and $\psi(\dots)$ of $9n+3l$ arguments of the ring language such that for decompositions $x_1 = u_1 t_1 v_1 u_1'$ and $x_2 = u_2 t_2 v_2 u_2'$ and $x_3 = u_3 t_3 v_3 u_3'$, where

$$u_i = x_{\alpha_1}(t_1^{(i)}) \dots x_{\alpha_n}(t_n^{(i)}), \qquad u_i' = x_{\alpha_1}(s_1^{(i)}) \dots x_{\alpha_n}(s_n^{(i)}),$$
$$v_i = x_{-\alpha_1}(r_1^{(i)}) \dots x_{-\alpha_n}(r_n^{(i)}), \quad t_i = h_{\alpha_1}(\xi_1^{(i)}) \dots h_{\alpha_l}(\xi_l^{(i)}), \quad i = 1,2,3$$

the formula $\varphi(t_k^{(1)}, t_k^{(2)}, s_k^{(1)}, s_k^{(2)}, r_k^{(1)}, r_k^{(2)}, \xi_k^{(1)}, \xi_k^{(2)})$ holds iff $x_1 = x_2$ and $\psi(t_k^{(1)}, t_k^{(2)}, t_k^{(3)}, s_k^{(1)}, s_k^{(2)}, s_k^{(3)}, r_k^{(1)}, r_k^{(2)}, r_k^{(3)}, xi_k^{(1)}, \xi_k^{(2)}, \xi_k^{(3)})$ holds iff $x_1 = x_2 x_3$.

Sketch of the proof. Step 3

A Chevalley group $G_\pi(\Phi, R)$ (or its elementary subgroup $E_\pi(\Phi, R)$) **is bi-interpretable with the ring** R **with parameters** $\overline{x} = \{x_\alpha(1) \mid \alpha \in \Phi\}$.

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Remark

This step can be possible for all Chevalley groups with bounded generation (maybe wider).

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The properties of our parameters $\overline{x}$ can be expressed by a first order formula $\Phi(\overline{x})$, which means that there exists a ring $R' = (X_\alpha, \oplus, \otimes)$ such that $E_{\mathrm{ad}}(\Phi, R) \cong E_{\mathrm{ad}}(\Phi, R')$ (using Gauss decomposition with elementary formulas).

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By algebraic results (Bunina for these types of rings) an isomorphism is the composition of a ring isomorphism (where $x_\alpha(t) \mapsto x_\alpha(t')$ for all $\alpha \in \Phi$ and $t \in R$) and some automorphism of the initial group $E_{\text{ad}}(\Phi, R)$. This means that $R' \cong R$ and any parameters $\overline{t}$ satisfied the formula Φ up to an automorphism of the group $E_{\text{ad}}(\Phi, R)$ have the form $\overline{x} = \{x_\alpha(1) \mid \alpha \in \Phi\}$, what was required.

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Remark

This step can be possible for all Chevalley groups with bounded generation.

Conjectures

1. If for some type of rings elementary Chevalley groups over them have boundary generation and for them exists any analogue of Gauss decomposition with elementary equality and multiplication, then the same theorems holds (regular bi-interpretability and elementary definability) — almost sure.

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THANK YOU!