Twisted conjugacy in Chevalley groups

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Twisted conjugacy

If ϕ is an automorphism of G then G acts on itself by

$$g \cdot x = gx\phi(g^{-1}).$$

This action is called the *ϕ-twisted conjugacy* and the corresponding orbits are the *ϕ-twisted conjugacy classes*.

It is of interest to know whether the number of ϕ-twisted conjugacy classes in a given (infinite) group is infinite for every automorphism ϕ.

The R_∞-property

The number of ϕ-twisted conjugacy classes is denoted by $R(\phi)$ where $R(\phi) \in \mathbb{N} \cup \{\infty\}$. ....*R for Reidemeister*.

We say that a group G has the *R_∞-property* if $R(\phi)$ is infinite for every automorphism ϕ of G.

The group G is then called an R_∞-group.

Fel'shtyn-Hill (1994) initiated the problem of determining R_∞-groups. It has been an active area of research since then.

Known results

- Fel'shtyn and co-authors: 1994 – now

- Bhunia, Bose, Gonçalves, Mitra, Mubeena, Nasybullov, Sankaran, Tabac, Wong, ... (in the alphabetical order)

- Nasybullov (2012 till now):

$\mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$, $\mathrm{Sp}_{2n}(k)$ and $\mathrm{O}_n(k)$ are R_∞-groups where k is an integral domain of zero characteristic and $\mathrm{Aut}(k)$ is periodic.

Further, Chevalley groups over fields with finite transcendence degree over $\mathbb{Q}$.

- Mubeena-Sankaran (2014, 2016): $\mathrm{SL}_n(\mathbb{Z})$, $\mathrm{PSL}_n(\mathbb{Z})$, ... are R_∞-groups.

Further, irreducible lattices in connected, semisimple Lie groups G of real rank ≥ 2.

Known results

- Bhunia-Bose: $G(k)$, $k = \bar{k}$, $G =$ connected, non-solvable linear algebraic group.

Further, they studied some solvable groups in a recent work.

Only the algebraic automorphisms are considered by B-B.

- Mitra-Sankaran (2022): $\mathrm{GL}_2(R)$ and $\mathrm{SL}_2(R)$, $R = \mathbb{F}_q[t]$.

Further, $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R)$ were also shown to be R_∞-groups.

- What about other groups over such R?

We study all classical groups over such rings R.

Automorphisms of Chevalley groups

Let G be a Chevalley group over a ring R. We list some automorphisms of G.

- Inner automorphisms, $g \mapsto hgh^{-1}$,
- ring automorphisms, ρ,
- graph automorphisms, ϵ,
  only for A_n, D_n (especially D_4) and E_6,
- central automorphisms.

An automorphism ϕ of G is called *standard* if it is a composition of the above four types of automorphisms.

An important theorem

Theorem (Bunina, 2012)

If R is a commutative ring with $1/2 \in R$ and Φ is a classical irreducible root system of rank > 1 then every automorphism of the adjoint Chevalley group $G(\Phi, R)$ is standard.

Once we have information about automorphisms then we have to check if $R(\phi) = \infty$ for each automorphism ϕ.

Some basic tools

- We need only to look at a transversal of $\mathrm{Inn}(G)$ in $\mathrm{Aut}(G)$.

- For a finite index characteristic $K \triangleleft G$, if G/K is R_∞-group then so is G.

- (Abe-Hurley, 1988) If G is an adjoint Chevalley group of rank at least 2 over a commmutative ring R then $Z(G)$ is trivial.

- We therefore deal only with automorphisms of the type

$$\rho \circ \epsilon$$

where ρ is a ring automorphism and ϵ is a graph automorphism.

Main result

Theorem (Garge–Mitra, 2022)

Let F be a subfield of $\overline{\mathbb{F}}_p$ with $p \neq 2$.

Let $F[t] \subseteq R \subsetneq F(t)$.

Let G be a simple classical Chevalley group of adjoint type over R of rank at least 2.

Then G has the R_∞-property.

THANK YOU!