Non-alternating Hamiltonian Lie algebras

A.V. Kondrateva and M.I Kuznetsov

Abstract. The general theory of non-alternating Hamiltonian Lie algebras over a perfect field of characteristic two is discussed. The complex of symmetric differential forms over an algebra of divided powers is constructed. The classification of graded algebras is given and their filtered deformations are found. The classes of equivalence of non-alternating Hamiltonian forms with polynomial coefficients in divided powers are described.

Introduction

Well known Lie algebras of Cartan type consisting of vector fields preserving special, Hamiltonian or contact form have the analogous in characteristic p. Note that the algebra of functions must be replaced by an algebra of divided powers $\mathcal{O}_n(\mathcal{F})$ (see [1]) that corresponds to some generalized flag $\mathcal{F}$ of the space E, $\mathcal{F} : E = E_0 \supseteq E_1 \supseteq \ldots$. Let $\{x_1, \ldots, x_n\}$ be a basis of E coordinated with $\mathcal{F}$ then $\mathcal{O}_n(\mathcal{F}) = \mathcal{O}(n : \mathbf{m})$ where $\mathbf{m} = (m_1, \ldots, m_n)$ is the n-tuple of heights of variables $x_1, \ldots, x_n$ with respect to the flag $\mathcal{F}$. In the case of a field of characteristic 2 one can construct a large class of simple Hamiltonian Lie algebras corresponding to non-alternating symmetric differential forms.

The class of non-alternating Hamiltonian Lie algebras may present interest for, at least, two reasons. First, it can be attributed to algebras of Cartan type when $p = 2$, which is of great importance for the classification of simple Lie algebras of low characteristics. Second, these Lie algebras may admit non-singular derivations, thus may be of interest for p-group theory.

Non-alternating Hamiltonian algebras over a field of characteristic 2 were first constructed in 1993 by Lin Lei [2] as Lie algebras of polynomials in divided powers with the symmetric Poisson bracket $\{f, g\} = \sum_i \partial_i f \partial_i g$. In the case when the heights of the variables are equal to 1, non-alternating Hamiltonian Lie algebras are isomorphic to the first series of simple Lie algebras built by I. Kaplansky [3]. In [4, 5], symmetric differential forms in divided powers are introduced and non-alternating Hamiltonian Lie algebras similar to Hamiltonian Lie superalgebras of characteristic zero with respect to the standard Poisson brackets are studied. Note that the classification of alternating Hamiltonian forms over an algebra of truncated polynomials is obtained by M.I. Kuznetsov, S.A. Kirillov ([6]). The complete classification over a divided powers algebra is built by S.M. Skryabin ([7, 8]).

In the talk, the general theory of non-alternating Hamiltonian Lie algebras in divided powers over a perfect field of characteristic 2 is discussed. The main results may be found in [9]. In [10] the invariant construction of the complex of symmetric differential forms in characteristic 2 was given and some program of investigation was proposed. In the first stage the authors have obtained all invariants of symmetric Hamiltonian differential forms with constant coefficients with respect to parabolic subgroup of $GL(V)$ corresponding to flag $\mathcal{F}$. In particular, it was shown that there exists a basis of V coordinated with flag $\mathcal{F}$ such that a form has a matrix $diag(M_0, \ldots, M_0,\ M_1, \ldots, M_1,\ 1_s)$ where

$$M_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Later on, employing the theory of truncated coinduced modules ([11, 12]) the authors have proved that in the case when $n > 4$ or $n = 4$ and the flag $\mathcal{F}$ is non-trivial or $n = 2,\ 3$ and E_1 contains a non-isotropic vector with respect to the form $\overline{\omega}$ on E dual to the form ω, each filtered Lie algebra associated with a graded non-alternating Hamiltonian algebra is given by a non-alternating Hamiltonian differential form with non-constant polynomial coefficients in divided powers.

One of the main results is the proof of the equivalence of the non-alternating Hamiltonian form ω with polynomial coefficients to its initial form $\omega(0)$, provided that the canonical form of $\omega(0)$ contains $(dx_i)^{(2)}$ or $dx_i dx_j + (dx_j)^{(2)}$ for some variable x_i of the height greater than 1. The authors present some results on the problem of the classification of non-alternating Hamiltonian forms for which this condition is not fulfilled. In this case the non-alternating Hamiltonian form ω may not be equivalent to $\omega(0)$.

Theorem. *Let ω be a non-alternating Hamiltonian form over the algebra of divided powers $\mathcal{O}_n(\mathcal{F})$, E^0 be the subspace of all isotropic vectors of E with respect to form $\overline{\omega(0)}$ on E dual to $\omega(0)$.*
(i) if $E_1 \not\subset E^0$ then ω is equivalent to $\omega(0)$;
(ii) if $E_1 \subset E^0$ then there exists a basis of E coordinated with $\mathcal{F}$ such that ω is equivalent to

$$dx_1 dx_2 + \ldots + dx_{n-2} dx_{n-1} + dx_n^{(2)} + \sum_{j=1}^{n-1} b_{jn} dz_j dz_n \quad \textit{if } n{=}2k{+}1$$

and

$$dx_1 dx_2 + \ldots + dx_{n-1} dx_n + dx_n^{(2)} + \sum_{j \neq n-1} b_{j,n-1} dz_j dz_{n-1} \quad \textit{if } n{=}2k$$

Moreover, if $b_{jn} \neq 0$ ($b_{j,n-1} \neq 0$) for some j then ω is not equivalent to $\omega(0)$.

The problem of classification of forms with polynomial (in divided powers) coefficients has been solved completely only in the case of three variables [13]. In addition, the dimensions of all simple non-alternating Hamiltonian Lie algebras are found.

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A.V. Kondrateva
Institute of Information Technology, Mathematics and Mechanics
Nizhny Novgorod State University
Nizhny Novgorod, Russia
e-mail: alisakondr@mail.ru

M.I Kuznetsov
Institute of Information Technology, Mathematics and Mechanics
Nizhny Novgorod State University
Nizhny Novgorod, Russia
e-mail: kuznets-1349@yandex.ru