Orbits of Algebraic Groups and Classification Problems

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International Conference
Algebraic Groups, Their Friends and Relations
September 21, 2022
St. Petersburg Department of Steklov Mathematical Institute
Russian Academy of Sciences

Notation

Notation

- k an algebraically closed ground field of arbitrary characteristic.
- G a connected linear algebraic group.
- V a finite-dimensional algebraic kG-module. The action of G of V is denoted as

$$G \times V \to V, \quad (g, v) \mapsto g \cdot v.$$

Main problem

The following problem permanently arises in algebraic transformation group theory and its applications to various classification problems:

Problem ($*$)

Given two points a and $b \in V$, how can one find out whether or not they lie in the same G-orbit?

Decidability

Elucidation

By "finding out" we mean **decidability** of Problem $(*)$, i.e., the existence of an **algorithm** providing a correct yes-or-no answer to Problem $(*)$ by means of finitely many effectively feasible operations.

Warning

Finding an algorithm with the best parameters (running time and memory used) is a **separate topic** lying beyond the scope of this talk.

Example 1: Classification of algebras

L a finite dimensional vector space over k.
$V = L^* \otimes L^* \otimes L$.

Points of V are the **structures of (not necessarily associative) k-algebras on the vector space L**:

$v = \sum f \otimes h \otimes \ell \in V = L^* \otimes L^* \otimes L$ determines **the algebra $\mathcal{A}(v)$** with the multiplication

$$st := \sum f(s)h(t)\ell \quad \text{for all } s, t \in L.$$

$G = \mathrm{GL}(L)$ naturally acts on $V = L^* \otimes L^* \otimes L$.

$\mathcal{A}(a)$ and $\mathcal{A}(b)$ are isomorphic algebras $\iff$ $G \cdot a = G \cdot b$.

Example 1: Classification of algebras (continued)

If $G \cdot a$ lies in the closure of $G \cdot b$, then $\mathcal{A}_a$ is called a **degeneration** of $\mathcal{A}_b$.

In general case, to find out whether $\mathcal{A}_a$ is a degeneration of $\mathcal{A}_b$ is considered a difficult problem. In some special cases degeneration are classified by means of ad hoc methods.

Example 2: Classification of modules

L a finite dimensional vector space over k.
A an associative k-algebra.
Every k-algebra homomorphism $\varphi\colon A \to \mathrm{End}(L)$ determines an A-module structure on L by

$$a \cdot \ell := \varphi(a)(\ell) \quad \text{for all } a \in A,\ \ell \in L.$$

This yields a bijection between the sets of such homomorphisms and A-module structures.
In turn, if A is generated by the elements $a_1, \ldots, a_d$, then φ is determined by the point

$$s = \big(\varphi(a_1), \ldots, \varphi(a_d)\big) \in V = \mathrm{End}(L) \times \cdots \times \mathrm{End}(L) \quad (d \text{ factors}).$$

Denote $\mathcal{M}_s$ the A-module corresponding to this φ.

Example 2: Classification of modules (continued)

When φ runs over all homomorphisms, s runs over a closed algebraic subset $\mathrm{Mod}_A(L)$ in V called the **variety of A-module structures on L**.

$G = \mathrm{GL}(L)$ diagonally acts on V by conjugation.

$\mathcal{M}_a$ and $\mathcal{M}_b$ are isomorphic A-modules $\iff$ $G \cdot a = G \cdot b$.

Example 2: Classification of modules (continued)

If $G \cdot a$ lies in the closure of $G \cdot b$, then $\mathcal{M}_a$ is called a **degeneration** of $\mathcal{M}_b$.

In general case, to find out whether $\mathcal{M}_a$ is a degeneration of $\mathcal{M}_b$ is considered a difficult problem. In some cases it is solved by means of ad hoc methods (for instance, if A is the path algebra of an oriented extended Dynkin graph of a root system of type A_l, D_l, E_6, E_7, or E_8 (Bongartz, 1995)).

Example 3: Classification of representations of quivers

Q a quiver (i.e., a finite directed graph).
$v_1, \ldots, v_m$ the set of all vertices of Q.
E the set of all edges of Q.
$t\alpha$ (resp. $h\alpha$) the tail (resp. head) of $\alpha \in E$.
U_{v_i} a finite-dimensional vector space over k assigned to every v_i.
The k-vector space

$$V = \bigoplus_{\alpha \in E} \mathrm{Hom}_k(U_{t\alpha}, U_{h\alpha}).$$

is the set of all representations of Q of dimension $\big(\dim U_{v_1}, \ldots, \dim U_{v_m}\big) \in \mathbb{N}^m$.

Example 3: Classification of representations of quivers (continued)

The group

$$G = \mathrm{GL}(U_{v_1}) \times \cdots \times \mathrm{GL}(U_{v_m})$$

acts on V by

$$(g_{v_1}, \ldots, g_{v_n}) \cdot \bigoplus_{\alpha \in E} h_{t\alpha, h\alpha} = \bigoplus_{\alpha \in E} g_{h\alpha} h_{t\alpha, h\alpha} g_{t\alpha}^{-1}.$$

Representations $a, b \in V$ are equivalent $\iff$ $G \cdot a = G \cdot b$.

Example 4: Classification of smooth hypersurfaces

h_1 and h_2 irreducible forms of the same positive degree in the homogeneous coordinates of $\mathbb{P}^n$.

H_i the hypersurface in $\mathbb{P}^n$ defined by the equation $h_i = 0$.

Theorem (Severy–Lefschetz–Andreotti)

Every positive divisor on any smooth hypersurface in $\mathbb{P}^n$, where $n \geqslant 4$, is cut out by a hypersurface in $\mathbb{P}^n$.

This implies:

If $n \geqslant 4$ and both H_1 and H_2 are smooth, then

H_1 and H_2 are isomorphic varieties $\iff \mathrm{GL}_{n+1} \cdot h_1 = \mathrm{GL}_{n+1} \cdot h_2$.

Example 5: Classification of algebraic curves of genus $g \geqslant 2$

There are other types of algebraic varieties for which the isomorphism problem is reduced to finding out whether some forms lie in the same orbit of a linear algebraic group. For instance, this is so for algebraic curves.

Theorem

- *Every irreducible smooth projective curve of a genus $g \geqslant 2$ is embedded into $\mathbb{P}^{5g-6}$ be means of its tripled canonical class.*
- *The following properties are equivalent:*
  - *Two such curves X and Y are isomorphic.*
  - *Their images $\widetilde{X}$ and $\widetilde{Y}$ in $\mathbb{P}^{5g-6}$ are transformed one into another by a projective transformation of $\mathbb{P}^{5g-6}$.*
  - *The Chow forms of $\widetilde{X}$ and $\widetilde{Y}$ lie in the same orbit of the corresponding GL_N.*

Main problem

The key property of orbits of algebraic group actions related to Problem $(*)$ is given by the following classical

Theorem

Every orbit of algebraic group action is open in its closure.

This yields

Corollary

$$G \cdot a = G \cdot b \iff G \cdot a \subseteq \overline{G \cdot b} \text{ and } G \cdot b \subseteq \overline{G \cdot a}.$$

Main problem

This leads to a more general

Problem $(**)$

Given two points a and $b \in V$*, how can one find out whether or not* $G \cdot a$ *lies in* $\overline{G \cdot b}$*?*

Example: Orbit closures and Geometric Complexity Theory

Using the sizes of arithmetic circuits computing multivariate polynomials over $k = \mathbb{C}$, one defines certain classes **VP** and **VNP** $\supseteq$ **VP** of sequences of such polynomials.

Valiant's conjecture:
VP $\neq$ **VNP**

This conjecture can be seen as an algebraic version of Cook's famous $P \neq NP$ hypothesis.

Example: Orbit closures and Geometric Complexity Theory (continued)

The following major result led to a conjecture about orbit closures that implies Valiant's conjecture.
Let

$$\mathrm{perm}_n := \text{permanent of} \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \ldots & \ldots & \ldots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \text{ where } x_{ij} \text{ are variables}$$

Theorem (Valiant)

The following properties are equivalent:

- *The sequence* $\mathrm{perm}_1, \mathrm{perm}_2, \ldots$ *does* <u>*not*</u> *lie in* **VP**.
- **VP** $\neq$ **VNP**.

Example: Orbit closures and Geometric Complexity Theory (continued)

M_m the vector space of $m \times m$-matrices with coefficients in k.
$x_{i,j}$ the standard coordinate functions on M_m.
n a positive integer, $n < m$.

$$\begin{aligned} d_m\colon \mathrm{M}_m \to k \quad & A \mapsto \det A, \\ p_n\colon \mathrm{M}_m \to k \quad & A \mapsto x_{1,1}^{m-n} \cdot \text{permanent of } A_n, \end{aligned}$$

where A_n is the right down $n \times n$-corner of A.
GL_{m^2} naturally acts on the space of polynomial functions $\mathrm{M}_m \to k$.

Theorem

There is n_0 such that for all $n \geqslant n_0$ and m large enough compared to n, namely, $m = O(n^2 2^n)$,

$$p_n \text{ lies in the closure of } \mathrm{GL}_{m^2} \cdot d_m. \qquad (\mathbf{!})$$

Example: Orbit closures and Geometric Complexity Theory (continued)

Let $m(n)$ be the **minimal** m such that property (!) holds for every $n \geqslant n_0$.

Conjecture (Mulmuley, Sohoni)

The function $n \mapsto m(n)$ grows faster than any polynomial in n.

Theorem

Mulmuley–Sohoni's conjecture $\Rightarrow$ Valiant's conjecture.

Plan

The talk is aimed to explain that

Problem $(**)$ is decidable if $\operatorname{char} k = 0$ and G is reductive.

The plan:

- Describing the input of the algorithm.
- Reducing to the case of conical orbits $G \cdot a$ and $G \cdot b$.
- Describing the algorithm.

Describing the input of the algorithm

The input of the algorithm consists of **two components**.

The **first component** is a certain
set of n^2 functions describing the G-action on V.

This set is constructed using the following

rational parametrization of G.

On the input: Rational parametrization of G

Notation

- $\mathbb{A}^{r,s} := \{(\varepsilon_1, \ldots, \varepsilon_{r+s}) \in \mathbb{A}^{r+s} \mid \varepsilon_1 \cdots \varepsilon_r \neq 0\}, \qquad r, s \in \mathbb{N}.$
- $x_1, \ldots, x_{r+s}$ the standard coordinate functions on $\mathbb{A}^{r,s}$:

$$x_i\big((\varepsilon_1, \ldots, \varepsilon_{r+s})\big) = \varepsilon_i.$$

The (ordered) set of all Laurent monomials

$$x_1^{i_1} \cdots x_{r+s}^{i_{r+s}}, \quad \text{where} \quad i_1, \ldots, i_r \in \mathbb{Z} \text{ and } i_{r+1}, \ldots, i_{r+s} \in \mathbb{N},$$

is a basis of the k-vector space

$$k[\mathbb{A}^{r,s}] = k[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}, x_{r+1}, \ldots, x_{r+s}]$$

of all regular functions on $\mathbb{A}^{r,s}$.

On the input: Rational parametrization of G

Lemma

There is an open embedding

$$\iota\colon \mathbb{A}^{r,s} \hookrightarrow G$$

with $r = \operatorname{rk} G$.

On the input: Example of rational parametrization

Example

Let $G = \mathrm{SL}_2$.
The map

$$\iota\colon \mathbb{A}^{1,2} \to \mathrm{SL}_2,$$

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mapsto \begin{bmatrix} 1 & \varepsilon_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \varepsilon_3 & 1 \end{bmatrix} = \begin{bmatrix} \varepsilon_1^{-1}\varepsilon_2\varepsilon_3 + \varepsilon_1 & \varepsilon_1^{-1}\varepsilon_2 \\ \varepsilon_1^{-1}\varepsilon_3 & \varepsilon_1^{-1} \end{bmatrix}$$

is an open embedding.

On the input: Data determining G-action on V

Notation

$e_1, \ldots, e_n$ a basis in V and

$$\rho_{i,j} \colon G \to k, \quad i, j \in \{1, \ldots, n\}$$

the regular functions on G (**matrix coefficients**) such that the action of G on V is given in the basis $e_1, \ldots, e_n$ by the matrix representation

$$\rho \colon G \to \mathrm{GL}_n, \quad \rho(g) = \begin{bmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,n}(g) \\ \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots} \\ \rho_{n,1}(g) & \cdots & \rho_{n,n}(g) \end{bmatrix}, \quad g \in G,$$

On the input: Data determining G-action on V

In other words,

$$g \cdot \Big(\sum_{i=1}^{n} \gamma_i e_i\Big) = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \rho_{j,i}(g)\gamma_i\Big) e_j \quad \text{for all } g \in G \text{ and } \gamma_i \in k.$$

On the input: Classical example of binary forms

The next example is the main object of the pre-Hilbertian classical invariant theory.

Example

$\mathrm{char}\, k = 0$.
$G = \mathrm{SL}_2$.
$V = \mathscr{B}_h$ the $(h+1)$-dimensional space of binary forms of degree h in variables z_1, z_2 over k with the G-action given by

$$g \cdot z_1 = \alpha z_1 + \gamma z_2, \ \ g \cdot z_2 = \beta z_1 + \delta z_2 \qquad \text{for } g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in G.$$

As a basis $e_1, \ldots, e_n$ (with $n = h+1$) in $\mathscr{B}_h$ take

$$e_i = z_1^{h+1-i} z_2^{i-1}.$$

Example (continued)

Then

$$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\cdot e_j = (\alpha z_1+\gamma z_2)^{h+1-j}(\beta z_1+\delta z_2)^{j-1}$$
$$= \sum_{i=1}^{h+1}\rho_{i,j}(g) z_1^{h+1-i} z_2^{i-1}$$

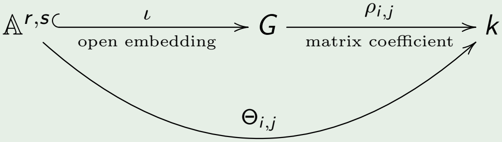
For instance, if $h=2$, then

$$\rho_{2,3}\colon \mathrm{SL}_2\to k,\quad \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\mapsto 2\beta\delta,$$
$$\rho_{2,2}\colon \mathrm{SL}_2\to k,\quad \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\mapsto \alpha\delta+\gamma\beta$$

The first component of the input

The first component of the input is the following system of n^2 regular functions on $\mathbb{A}^{r,s}$:

$$\Theta_{i,j} := \rho_{i,j} \circ \iota \in k[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}, x_{r+1}, \ldots, x_{r+s}]$$



The first component of the input: Classical example of binary forms

Example

$\mathrm{char}\, k = 0$, $G = \mathrm{SL}_2$,
ι, $V = \mathscr{B}_h$, $e_i = z_1^{h+1-i} z_2^{i-1}$ as in the above examples.

Then $\Theta_{i,j}$ is the coefficient of $z_1^{h+1-i} z_2^{i-1}$ in the decomposition of

$$\big((x_1 + x_1^{-1}x_2x_3)z_1 + (x_1^{-1}x_3)z_2\big)^{h+1-j}\big((x_1^{-1}x_2)z_1 + (x_1^{-1})z_2\big)^{j-1}$$

as a linear combination of monomials in z_1, z_2 with the coefficients in $k[x_1, x_1^{-1}, x_2, x_3]$.

The first component of the input: Classical example of binary forms

Example (continued)

For instance, if $h = 2$, then

$$\Theta_{2,2}\colon \mathbb{A}^{1,2} \to k,$$
$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mapsto \alpha\delta + \gamma\beta\Big|_{\substack{\alpha=\varepsilon^{-1}\varepsilon_2\varepsilon_3+\varepsilon_1,\\ \beta=\varepsilon_1^{-1}\varepsilon_2,\\ \gamma=\varepsilon_1^{-1}\varepsilon_3,\\ \delta=\varepsilon_1^{-1}}} = 1 + 2\varepsilon_1^{-2}\varepsilon_2\varepsilon_3.$$

Therefore,

$$\Theta_{2,2} = 1 + 2x_1^{-2}x_2x_3.$$

The second component of the input: deg $\rho(G)$

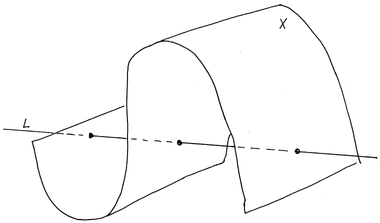
X a locally closed irreducible subset of $\mathbb{A}^m$.
$\mathscr{L}$ the set of all $(m - \dim X)$-dimensional linear subspaces of $\mathbb{A}^m$.
$\mathscr{L}$ has a structure of irreducible algebraic variety.

We use the following classical

Theorem

For every L in a dense open subsubset of $\mathscr{L}$, the intersection $X \cap L$ is a finite set whose cardinality does not depend on L.

This cardinality is called the **degree of** X and denoted by $\deg \boldsymbol{X}$.



The second component of the input: deg $\rho(G)$

As $\rho(G)$ is a locally closed irreducible subset of the n^2-dimensional affine space $\mathrm{Mat}_{n,n}(k)$, the integer $\deg \rho(G)$ is defined.

The integer $\deg \rho(G)$ is the second component of the input of the algorithm.

The reason to consider $\deg \rho(G)$ known is that

if $\mathrm{char}\, k = 0$ and G is **reductive**, then there is **a formula for $\deg \rho(G)$** .

In this formula, the following notation is used.

Formula for $\deg \rho(G)$: Notation

Notation

- T a maximal torus of G.
- $r = \dim T = \mathrm{rk}\, G$.
- $X(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ in additive notation and naturally embedded in $E = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
- E and $\mathbb{R}^r$ are identified by means of an isomorphism $E \xrightarrow{\simeq} \mathbb{R}^r$ such that $X(T)$ is identified with $\mathbb{Z}^r$.
- $d\nu$ the standard volume form on E.
- $W := N_G(T)/T \subset \mathrm{GL}(E)$ the Weyl group.
- $\mathcal{P}_V \subset E$ the convex envelope of 0 and all T-weights of V.

Formula for $\deg \rho(G)$: Notation

Notation (continued)

- We assume $\dim \mathcal{P}_V = r$. This is attained by replacing G with $G/(\ker \rho)^0$.
- R_+ a system of positive roots of G with respect to T.
- $\alpha^\vee$ the coroot corresponding to a root $\alpha \in R_+$, i.e.,

$$\alpha^\vee : E \to \mathbb{R}, \quad v \mapsto 2\langle \alpha \,|\, v\rangle / \langle \alpha \,|\, \alpha\rangle,$$

  where $\langle \ | \ \rangle$ is a W-invariant inner product on E.
- $m_1, \ldots, m_r$ the exponents of W (i.e., $m_1 + 1, \ldots, m_r + 1$ are the degrees of homogeneous free generators of $\mathbb{R}[E]^W$).

Particular case: $G = T$. Then

$$R_+ = \varnothing, \quad |W| = 1, \quad m_1 = \cdots = m_r = 0.$$

Formula for $\deg \rho(G)$: Theorem

Theorem (B. Kazarnovskiĭ)

Let $\operatorname{char} k = 0$*, let* G *be reductive, and let* $\ker \rho$ *be finite. Then*

$$\deg \rho(G) := \frac{\dim G\,!}{|W|(m_1! \cdots m_r!)^2 |\ker \rho|} \int_{\mathcal{P}_V} \prod_{\alpha \in R_+} (\alpha^\vee)^2 d\nu.$$

Formula for $\deg \rho(G)$: Classical example of binary forms

Example

char $k = 0$, $G = \mathrm{SL}_2$, $V = \mathscr{B}_h$ the kG-module of binary forms of degree h in variables z_1, z_2.
Then

$$\dim G = 3, \quad r = 1, \quad E = \mathbb{R}, \quad X(T) = \mathbb{Z}, \quad |W| = 2, \quad m_1 = 1,$$
$$R_+ = \{\alpha = 2\},$$
$$\alpha^\vee \text{ is the standard coordinate function } x\colon \mathbb{R} \to \mathbb{R}, \ a \mapsto a,$$
$$\ker_V = \begin{cases} \text{trivial} & \text{if } h \text{ is odd,} \\ \text{cyclic group } \big(\mathrm{diag}(-1,-1)\big) \text{ of order } 2 & \text{if } h \text{ is even.} \end{cases}$$

Formula for $\deg \rho(G)$: Classical example of binary forms

Example (continued)

Take
$$T = \{\mathrm{diag}(t, t^{-1}) \mid t \in k \setminus \{0\}\}.$$
Then $e_i = z_1^{h+1-i} z_2^{i-1}$ is the T-weight vector of the weight $h - 2i + 2$. Hence $\{h, h-2, \ldots, -h+2, -h\}$ is the T-weight system of V. Whence
$$\mathcal{P}_V = [-h, h].$$
This yields
$$\deg \rho(\mathrm{SL}_2) = \frac{3!}{2|\ker \rho|} \int_{-h}^{h} x^2 dx = \begin{cases} 2h^3 & \text{if } h \text{ is odd,} \\ h^3 & \text{if } h \text{ is even.} \end{cases}$$

Reducing Problem $(**)$ to the case of conical orbits

Now we explain that when searching for an algorithmic solution to

Problem $(**)$

Given two points a and $b \in V$, how can one find out whether or not $G \cdot a$ lies in $\overline{G \cdot b}$?

one can assume that the orbits $G \cdot a$ and $G \cdot b$ are **conical**.

Definition

A subset C of a vector space L over k is called **conical** if it is stable with respect to scalar multiplication by every nonzero element of k:

$$\lambda C = C \quad \text{for every } \lambda \in k \setminus \{0\}.$$

Reducing Problem $(**)$ to the case of conical orbits: Step 1

The reduction is performed in **two steps**.

Step 1

Consider the G-action on the projective space $\mathbb{P}^n$ given by

$$g \cdot (\alpha_0 : \alpha_1 : \cdots : \alpha_n) := \left(\alpha_0 : \sum_{i=1}^{n} \rho_{1,i}(g)\alpha_i : \cdots : \sum_{i=1}^{n} \rho_{n,i}(g)\alpha_i\right).$$

Reducing Problem $(**)$ to the case of conical orbits: Step 1

This yields

the group embedding $\rho(G) \hookrightarrow \mathrm{Aut}\,\mathbb{P}^n$

and

the G-equivariant embedding of varieties $V \hookrightarrow \mathbb{P}^n$

whose image is the standard principal open subset

$$\{(\alpha_0 : \alpha_1 : \cdots : \alpha_n) \mid \alpha_0 \neq 0\}.$$

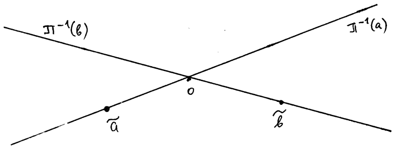
Reducing Problem $(**)$ to the case of conical orbits: Step 2

Consider the natural surjections

$$\tau\colon \mathrm{GL}_{n+1} \to \mathrm{Aut}\,\mathbb{P}^n \qquad \text{and} \qquad \pi\colon k^{n+1} \setminus \{0\} \to \mathbb{P}^n.$$

Step 2 Replace

- G with $\widetilde{G} := \tau^{-1}(\rho(G))$,
- V with $\widetilde{V} := k^{n+1}$,
- a and b with any $\widetilde{a}, \widetilde{b} \in \widetilde{V}$ such that $\pi(\widetilde{a}) = a, \pi(\widetilde{b}) = b$.



$\widetilde{G}$ is connected. $\widetilde{G}$ is reductive if G is.

Reducing Problem $(**)$ to the case of conical orbits

The following lemma shows that performing Steps 1 and 2 reduces Problem $(**)$ to the case of conical orbits.

Lemma

The following properties are equivalent:

- $G \cdot a$ *lies in the closure of* $G \cdot b$ *in* $\mathbb{P}^n$;
- $G \cdot a$ *lies in the closure of* $G \cdot b$ *in* V;
- $\widetilde{G} \cdot \widetilde{a}$ *lies in the closure of* $\widetilde{G} \cdot \widetilde{b}$ *in* $\widetilde{V}$.

The orbits $\widetilde{G} \cdot \widetilde{a}$ *and* $\widetilde{G} \cdot \widetilde{b}$ *are conical.*

Application to a more general setting

As every normal quasiprojective G-variety can be equivariantly embedded in some $\mathbb{P}^m$, it frequently arises the problem analogous to Problem $(**)$ but for a G-action on some **projective space**.

The above lemma shows that **this problem is reduced to Problem $(**)$ for linear actions on vector spaces**.

Algorithm for solving Problem $(**)$

We recall that

$$n = \dim V, \quad d = \deg \rho(G), \quad r = \mathrm{rank}\, G, \quad s = \dim G - r,$$

$$\Theta_{i,j} \in k[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}, x_{r+1}, \ldots, x_{r+s}]$$

(the restrictions of matrix coefficients of ρ to the open subset $\iota(\mathbb{A}^{r,s})$ of G)

The cases $n = 1$ and $\overline{G \cdot b} = V$ (i.e., $\dim G \cdot b = n$) are clear. Therefore, in what follows we assume

$$n \geqslant 2 \qquad \text{and} \qquad \dim G \cdot b < n.$$

Algorithm

Step 1

Following the procedure described above, reduce Problem $(**)$ to the case of **conical orbits** $G \cdot a$ and $G \cdot b$.

Algorithm

Step 2

Find the coordinates of a and b in the basis $e_1, \ldots, e_n$:

$$a = \alpha_1 e_1 + \cdots + \alpha_n e_n, \quad b = \beta_1 e_1 + \cdots + \beta_n e_n,$$

and, replacing $e_1, \ldots, e_n$ by another basis if necessary, **ensure that**

$$\beta_1 \cdots \beta_n \neq 0.$$

Algorithm

Step 3

Consider the "generic" polynomials $F_1, \dots, F_n$ of degree $2d-2$ in the variables $y_1, \dots, y_n$,

$$F_s := \sum_{\substack{q_1,\dots,q_n \in \mathbb{N} \\ q_1+\dots+q_n \leqslant 2d-2}} c_{s,q_1,\dots,q_n} y_1^{q_1} \cdots y_n^{q_n}, \quad s = 1, \dots, n,$$

where the coefficients $c_{s,q_1,\dots,q_n}$ are **indeterminates** over k, and then put

$$H := (y_1 - \alpha_1)F_1 + \dots + (y_n - \alpha_n)F_n - 1.$$

Algorithm

Step 4

Replace every y_i in H with

$$\sum_{j=1}^{n} \beta_j \Theta_{i,j}.$$

The result of this is a sum

$$\sum_{(i_1,\ldots,i_{r+s})\in M} \ell_{i_1,\ldots,i_{r+s}} x_1^{i_1} \cdots x_{r+s}^{i_{r+s}},$$

where M is a finite subset of $\mathbb{Z}^r \times \mathbb{N}^s$ and **every $\ell_{i_1,\ldots,i_{r+s}}$ is a linear combination of $c_{s,q_1,\ldots,q_n}$'s with the coefficients in k.**

Algorithm

Step 5

Consider the following **system of <u>linear</u> equations in variables** $c_{s,q_1,\ldots,q_n}$ **with the coefficients in** k:

$$\ell_{i_1,\ldots,i_{r+s}} = 0 \quad \text{where } (i_1, \ldots, i_{r+s}) \text{ runs over } M.$$

Denote this system by $(\bigstar)$.

Algorithm: The data-driven decision rule

Theorem

The following properties are **equivalent***:*

- *the orbit* $G \cdot a$ **lies in the closure** *of the orbit* $\overline{G \cdot b}$*;*
- *the system* $(\bigstar)$ *of linear equations is* **inconsistent***.*

Algorithm: The data-driven decision rule

Remark

By the Kronecker–Capelli theorem, this yields the following **numerical criterion**:

Let A be the coefficient matrix of ($\bigstar$) and let $\widetilde{A}$ be the augmented matrix obtained from A by adding the column of free terms.
Then

$$G \cdot a \subseteq \overline{G \cdot b} \iff \operatorname{rank} A \neq \operatorname{rank} \widetilde{A}.$$

Example: Binary forms

Example

$G = \mathrm{SL}_2$, $V = \mathscr{B}_h$.

The number of variables $c_{s,q_1,\ldots,q_n}$ in system $(\bigstar)$ is

$$(h+1)\binom{2h^3+h-1}{h+1} \qquad \text{if } h \text{ is even,}$$

$$(h+1)\binom{4h^3+h-1}{h+1} \qquad \text{if } h \text{ is odd.}$$

Algorithm: Replacing d by a smaller integer

Remark

It can be shown that

$$d = \deg \rho(G) \geqslant \deg G \cdot v \text{ for every } v \in V$$

and that the algorithm and Theorem remain valid if d is replaced with $\deg G \cdot b$.
Therefore, if $\deg G \cdot b$ is known, this leads to decreasing the number of variables and equations in system ($\bigstar$).

In some cases the degrees of orbits indeed have been computed.

Example: Degrees of orbits of binary forms

Example

$G = \mathrm{SL}_2$, $V = \mathscr{B}_h$.
Every nonzero $v \in \mathscr{B}_h$ decomposes as

$$v = \ell_1^{s_1} \cdots \ell_m^{s_m},$$

with pairwize nonproportional linear forms $\ell_1, \ldots, \ell_m \in \mathscr{B}_1$.
If $m \geqslant 3$ and $h/s_i \geqslant 2$ for every i, then the G-stabilizer G_v of v is finite.
Then, according to Moser-Jauslin (1992),

$$|G_v| \deg G \cdot v = -2(m-1)h^3 - 4\sum_{i=1}^{m}(h-s_i)^3 + 3h^2\sum_{i=1}^{m}(h-s_i) + 3h\sum_{i=1}^{m}(h-s_i)(h-2s_i)$$

Example: Degrees of orbits of binary forms

Example (continued)

In particular, if all roots of v are simple, i.e., $m = h$ and $s_1 = \cdots = s_h = 1$, then

$$\deg G \cdot v = \frac{2h(h-1)(h-2)}{|G_v|}.$$

This formula can also be deduced from a calculation made by Enriques and Fano in 1897. This has been done in 1983 by Mukai and Umemura (with a gap fixed in 1992 by Moser-Jauslin).