Chow ring of $B\,\mathrm{SO}(2n)$ in characteristic 2

Nikita A. Karpenko

Abstract. For $n \geq 1$, let $\mathrm{SO}(2n)$ be the special orthogonal group given by the standard split nondegenerate $2n$-dimensional quadratic form over a field. The Chow ring $\mathrm{CH}(B\,\mathrm{SO}(2n))$ of its classifying space has been computed for the field of complex numbers in 2000 by R. Field. Arbitrary fields of characteristic $\neq 2$ have been treated, using a different method, in 2006 – by L. A. Molina Rojas and A. Vistoli. Using specialization from characteristic 0, we extend their computation to characteristic 2.

Let G be an affine algebraic group over a field F. The Chow ring $\mathrm{CH}(BG)$ of the classifying space of G, considered systematically for the first time in [8], is the ring of characteristic classes for G, where a characteristic class is a functorial assignment for any G-torsor over a smooth F-variety X of an element in the Chow ring $\mathrm{CH}(X)$ of X.

Example 1. Let G be the general linear group $\mathrm{GL}(d)$ for some $d \geq 1$. A G-torsor over a smooth variety X yields a rank d vector bundle E over X. For $i = 1, \dots, d$, its ith Chern class $c_i(E)$ is an elements of $\mathrm{CH}^i(X)$ defining a characteristic classes $c_i \in \mathrm{CH}^i(BG)$. By [8], $c_1, \dots, c_d$ are independent generators of the ring $\mathrm{CH}(BG)$ identifying it with the polynomial ring $\mathbb{Z}[c_1, \dots, c_d]$.

For arbitrary G, given a faithful representation $G \hookrightarrow \mathrm{GL}(d)$, the pull-back ring homomorphism $\mathrm{CH}(B\,\mathrm{GL}(d)) \to \mathrm{CH}(BG)$ transfers the Chern classes

$$c_1, \dots, c_d \in \mathrm{CH}(B\,\mathrm{GL}(d))$$

to $\mathrm{CH}(BG)$. Besides, evaluating the characteristic classes for G on the base of the G-torsor

$$\mathrm{GL}(d) \to \mathrm{GL}(d)/G,$$

we get a ring homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(\mathrm{GL}(d)/G)$.

Theorem 2 ([9, Theorem 5.1])**.** *For any $d \geq 1$ and any faithful G-representation $G \hookrightarrow \mathrm{GL}(d)$, the homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(\mathrm{GL}(d)/G)$ is surjective; its kernel is the ideal generated by $c_1, \dots, c_d$.*

Remark 3. Theorem 2 is useful in both directions. First of all, it describes the Chow ring of the quotient variety $\mathrm{GL}(d)/G$ in terms of $\mathrm{CH}(BG)$. On the other hand, any given generators of the ring $\mathrm{CH}(\mathrm{GL}(d)/G)$ can be lifted to $\mathrm{CH}(BG)$; any such lifts together with the Chern classes $c_1, \ldots, c_d$ generate the Chow ring $\mathrm{CH}(BG)$.

Example 4. For the orthogonal group $\mathrm{O}(d)$, given by the standard split nondegenerate d-dimensional quadratic form over a field (of arbitrary characteristic), and its standard representation $\mathrm{O}(d) \hookrightarrow \mathrm{GL}(d)$, the quotient $\mathrm{GL}(d)/\mathrm{O}(d)$ is an open subset in an affine space. It follows that $\mathrm{CH}(\mathrm{GL}(d)/\mathrm{O}(d)) = \mathbb{Z}$ and so the ring $\mathrm{CH}(B\mathrm{O}(d))$ is generated by $c_1, \ldots, c_d$. In characteristic $\neq 2$, the relations are: $2c_i = 0$ for every odd i, [8, §15]. In characteristic 2, the relations are: $c_i = 0$ for every odd i, [5, Appendix B].

Now let us consider the special orthogonal group $\mathrm{SO}(d)$. For odd d, since $\mathrm{O}(d) = \mu_2 \times \mathrm{SO}(d)$, the ring homomorphism $\mathrm{CH}(B\mathrm{O}(d)) \to \mathrm{CH}(B\,\mathrm{SO}(d))$, induced by the embedding $\mathrm{SO}(d) \hookrightarrow \mathrm{O}(d)$, is surjective. Its kernel is generated by c_1. (In characteristic 2, since $c_1 = 0$, the kernel is trivial.)

The case of even $d = 2n$ is much more difficult. The group $\mathrm{SO}(4)$ – the first nontrivial case – was done over the complex numbers in [7]. The group $\mathrm{SO}(2n)$ for arbitrary n – still over the complex numbers – has been treated in [2] (see also [3]). Over an arbitrary field of characteristic $\neq 2$, the ("same") answer was obtained (by a different method) in [6]. Besides of the Chern classes, the answer involves certain characteristic class $y \in \mathrm{CH}^n(B\,\mathrm{SO}(2n))$ constructed by Edidin and Graham:

Theorem 5 ([6])**.** *For $n \geq 1$, the group $\mathrm{SO}(2n)$, considered over a field of characteristic $\neq 2$, has the Chow ring $\mathrm{CH}(B\,\mathrm{SO}(2n))$ generated by the Chern classes $c_2, c_3, \ldots, c_{2n}$ together with the Edidin-Graham characteristic class y. The generators are subject to the following relations:*

$$y^2 = (-1)^n 2^{2n-2} c_{2n} \quad \textit{and} \quad 2c_i = 0 = c_i \cdot y \textit{ for every odd } i.$$

We prove the analogue of Theorem 5 for characteristic 2. Any given field F of characteristic 2 is the residue field of some characteristic 0 discrete valuation field K, [1, Proposition 5 of §2.3 and Proposition 1 of §2.1 in Chapter IX]. We write $\mathrm{SO}(2n)_K$ and $\mathrm{SO}(2n)_F$ for the special orthogonal group over the respective fields and consider the specialization ring homomorphism

$$\mathrm{CH}(B\,\mathrm{SO}(2n)_K) \to \mathrm{CH}(B\,\mathrm{SO}(2n)_F). \qquad (6)$$

To explain the definition of (6), note that by [8, Theorem 1.3], the ring $\mathrm{CH}(BG)$ for an affine algebraic group G over any field is approximated by algebraic varieties over the field. By [8, Remark 1.4] (see also [5, Example 4.1]), in the case of $G = \mathrm{SO}(2n)$ it is enough to consider varieties obtained by base change from smooth schemes over the integers. For such varieties, the specialization homomorphism is a ring homomorphism defined in [4, Example 20.3.1].

Theorem 7. *The specialization homomorphism* (6) *is surjective; its kernel is generated by the odd Chern classes $c_3, c_5, \ldots, c_{2n-1}$.*

Theorems 5 and 7 together yield

Corollary 8. *For the special orthogonal group* $\mathrm{SO}(2n)$*, where* $n \geq 1$*, considered over a field of characteristic* 2*, the Chow ring* $\mathrm{CH}(B\,\mathrm{SO}(2n))$ *is generated by the* even *Chern classes* $c_2, c_4, \ldots, c_{2n}$ *together with the specialization* $y \in \mathrm{CH}^n(B\,\mathrm{SO}(2n))$ *of the Edidin-Graham characteristic class. These generators are subject to the unique relation*

$$y^2 = (-1)^n 2^{2n-2} c_{2n}.$$

Surjectivity of the specialization homomorphism is the most subtle part of Theorem 7. By Theorem 2, since the Chern classes specialize to "themselves", it is equivalent to the surjectivity of specialization for the quotient variety $\tilde{X} := \mathrm{GL}(2n)/\mathrm{SO}(2n)$. Note that a posteriori, the latter specialization homomorphism turns out to be an isomorphism.

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Nikita A. Karpenko
Mathematical & Statistical Sciences
University of Alberta
Edmonton
CANADA
e-mail: `karpenko@ualberta.ca`, *web page*: `www.ualberta.ca/~karpenko`