Twin H4s in Quaternions, Amalgamations and Iwahori-Matsumoto Decompositions

(joint work with R. Moody)

In honor of the 70th birthday of Prof. Nikolai A. Vavilov

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Contents

0. Abstract and Basic Notation

We find two root systems of type H_4 in quaternions. They create some infinite root system Σ. We will study it using amalgamations and Iwahori-Matsumoto decompositions.

$$\tau = \frac{1+\sqrt{5}}{2}, \quad \tau' = \frac{1-\sqrt{5}}{2}, \quad p = 1+i, \quad q = 1-i$$

$$\begin{array}{rcccl}
 & & & \mathcal{R}' & = \mathbb{Z}[\tau, \tfrac{1}{2}, i\ ] = \mathbb{Z}[\tau, i, \tfrac{1}{p}] \\
 & & \nearrow & \uparrow & \\
\mathbb{Z}[\tau, i] = & \mathcal{D}' & & \mathcal{R} & = \mathbb{Z}[\tau, \tfrac{1}{2}] \\
 & \uparrow & \nearrow & & \\
\mathbb{Z}[\tau] = & \mathcal{D} & & &
\end{array}$$

1. Quaternions

Let $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the quaternion (division) algebra,

satisfying $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$.

For $x = (x_1, x_2, x_3, x_4) \in \mathbb{H}$,

put $\tilde{x} = (x_1, -x_2, -x_3, -x_4)$

and $||x|| = \sqrt{x\tilde{x}} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$.

Then, $(x, y) = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$ and

$\mathbb{H} = \mathbb{R}^4$ is a Euclidean space.

Let $\mathbb{H}_1 = \{x \in \mathbb{H} \mid ||x|| = 1\}$.

2. <u>Unitary Group $SU(2)$</u>

Let $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

Then, $M = \mathbb{R}E \oplus \mathbb{R}I \oplus \mathbb{R}J \oplus \mathbb{R}K$ is a realization of $\mathbb{H}$.

For $x = (x_1, x_2, x_3, x_4) \in \mathbb{H}_1$, we see

$$x_1E + x_2I + x_3J + x_4K = \begin{pmatrix} x_1 + x_2 i & x_3 + x_4 i \\ -x_3 + x_4 i & x_1 - x_2 i \end{pmatrix}$$

belongs to $SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}$.

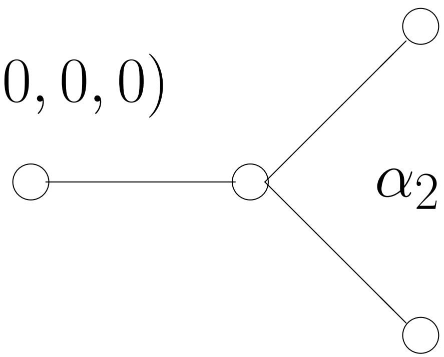
Therefore, $SU(2)$ is a realization of $\mathbb{H}_1$.

3. D_4 root system Δ''

Let

$$\Delta_0 = \{(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1)\},$$
$$\Delta_1 = \{\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) \mid \text{all signs}\} \text{ and}$$
$$\Delta'' = \Delta_0 \cup \Delta_1.$$

Then, Δ'' is a root system of type D_4 in $\mathbb{H}$.



$\alpha_1 = (1, 0, 0, 0)$

$\alpha_3 = (0, 1, 0, 0)$

$\alpha_2 = \frac{1}{2}(-1, -1, -1, -1)$

$\alpha_4 = (0, 0, 1, 0)$

4. H_4 Root Systems Δ, Δ'

Let $\Xi = \{(0, \pm 1, \pm\tau', \pm\tau) \mid \text{all signs, all even permutations}\}$ and

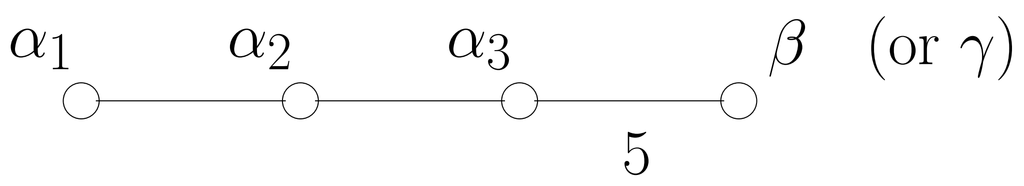
$$\Xi' = \{(0, \pm 1, \pm\tau, \pm\tau') \mid \text{all signs, all even permutations}\}.$$

Put $\Delta = \Delta'' \cup \Xi$, which is a root system of type H_4 in $\mathbb{H}$, and

$$\Delta' = \Delta'' \cup \Xi', \text{ which is another root system of type } H_4 \text{ in } \mathbb{H}.$$

Define $\alpha_1 = (1, 0, 0, 0), \quad \alpha_2 = \frac{1}{2}(-1, -1, -1, -1), \quad \alpha_3 = (0, 0, 1, 0),$

$$\beta = \frac{1}{2}(0, -\tau', -\tau, 1), \quad \gamma = \frac{1}{2}(0, 1, -\tau, -\tau'). \quad (\sharp\,\Delta = \sharp\,\Delta' = 120)$$



5. Infinite Root System Σ

Let $W = W(H_4),\ W' = W'(H_4),\ W'' = W''(D_4)$ be the Weyl groups of $\Delta,\ \Delta',\ \Delta''$, respectively.

Define $W^\infty = \langle W, W' \rangle \subset O(4, \mathcal{R})$ and

$$\Sigma = \{w(\alpha) \mid w \in W^\infty,\ \alpha \in (\Delta \cup \Delta')\}.$$

Known Facts. $\Delta, \Delta', \Delta''$ are subgroups of $\mathbb{H}_1$.

$$\begin{array}{ccccccccc} 1 & \to & \{\pm 1\} & \to & SU(2) & \to & SO(3) & \to & 1 \\ & & & & \bigcup & & \bigcup & & \\ & & & & \Delta, \Delta', \Delta'' & & A_5, A_4 & & \end{array}$$

Δ and Δ' are binary icosahedral groups, and Δ'' is a binary tetrahedral group.

6. Amalgamations

Put $SU(2,\mathcal{R}) = \left\{ \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \middle| \begin{array}{l} a,b,c,d \in \mathcal{R} \\ a^2+b^2+c^2+d^2 = 1 \end{array} \right\}.$

Theorem. [R. Moody-J.M.,2018]

(1) $\Sigma = \mathbb{H}_1 \cap \mathcal{R}^4 = SU(2,\mathcal{R}) = \Delta *_{\Delta''} \Delta'$ (amalgamation).

(2) $W \cap W' = N_W(\Delta'') = N_{W'}(\Delta'')$.

(3) $[W \cap W' : W''] = 3$.

(4) $W^\infty = W *_{W \cap W'} W'$ (amalgamation).

Note $W = \langle \sigma_{\alpha_1}, \sigma_{\alpha_2}, \sigma_{\alpha_3}, \sigma_\beta \rangle,\ W' = \langle \sigma_{\alpha_1}, \sigma_{\alpha_2}, \sigma_{\alpha_3}, \sigma_\gamma \rangle$ and
$W^\infty = \langle \sigma_{\alpha_1}, \sigma_{\alpha_2}, \sigma_{\alpha_3}, \sigma_\beta, \sigma_\gamma \rangle$.

7. Iwahori-Matsumoto Decompoisitions

Let A be a Euclidean domain, and $r \in A$ a nonzero prime element.

Suppose that the natural homomorphism $A^\times \to (A/rA)^\times$ is surjective.

Put $G = SL(2, A[\frac{1}{r}])$, and set

$$U_I = \left\langle \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ rs & 1 \end{pmatrix} \middle| s \in A \right\rangle,$$

$$T = \left\{ \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \middle| u \in A^\times \right\},$$

$$S = \left\{ w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, w_2 = \begin{pmatrix} 0 & -1/r \\ r & 0 \end{pmatrix} \right\}, \quad N = T\langle S \rangle,$$

$B_I = U_I T$ (Iwahori subgroup) and $W = N/T$ (Weyl group).

Theorem. [E.Abe-J.M.,1988]

(Also for Chevalley groups over Dedekind domains, in general.)

(1) (G, B_I, N, S) is a Tits system.

(2) $SL(2, A[\frac{1}{r}]) = \bigcup_{w \in W} B_I w B_I$ (Iwahori-Matsumoto decomposition).

(3) $W \simeq D_\infty$ (Infinite dihedral group, Affine Weyl group $A_1^{(1)}$).

Examples.

(1) $A = \mathbb{C}[t]$ and $r = t$. Then, $SL(2, \mathbb{C}[t, t^{-1}]) = \bigcup_{w \in W} B_I w B_I$.

(2) $A = \mathcal{D}'$ and $r = 1 + i$. Then, $SL(2, \mathcal{R}') = \bigcup_{w \in W} B_I w B_I$.

8. Additional Remarks (New Results)

Let $A = \mathcal{D}'$ and $r = p = 1 + i$. Then, $\mathcal{R}' = \mathcal{D}'[\frac{1}{p}]$.

Note $\Sigma = SU(2, \mathcal{R}) \subset SL(2, \mathcal{R}') = \bigcup_{w \in W} B_I w B_I$.

For each $w \in W$, we need to check $\Sigma \cap (B_I w B_I) \neq \emptyset$ or not.

Put

$$\mathcal{X}_n = \left\{ \begin{pmatrix} a/p^n & b/q^n \\ -\bar{b}/p^n & \bar{a}/q^n \end{pmatrix} \middle| \, a, b \in \mathcal{D}', |a|^2 + |b|^2 = 2^n, \gcd(a, b) = 1 \right\}.$$

[R.Moody-J.M.,2022]

(1) $\Sigma = \bigcup_{n\geq 0} \mathcal{X}_n$ (disjoint).

(2) $\Sigma \cap (B_I w B_I) \neq \emptyset \Leftrightarrow w \in \{1, (w_1 w_2)^n w_1 \mid n \geq 0\}$.

(3) $\mathcal{X}_0 = \Sigma \cap (B_I \cup B_I w_1 B_I) = \Sigma \cap SL(2, \mathcal{D}')$.

(4) $\mathcal{X}_n = \Sigma \cap \left(B_I (w_1 w_2)^n w_1 B_I\right)$ for $n \geq 1$.

(5) $\sharp\,\mathcal{X}_0 = 8$, $\sharp\,\mathcal{X}_1 = 16$, $\sharp\,\mathcal{X}_n = 3 \cdot 2^{2n+2}$ for $n \geq 2$.

$SL(2, \mathcal{R}') = \bigcup_{w \in W} B_I w B_I$	$B_I \cup B_I w_1 B_I$	$B_I (w_1 w_2)^n w_1 B_I$
$SU(2, \mathcal{R}) = \bigcup_{n \geq 0} \mathcal{X}_n$	$\mathcal{X}_0$	$\mathcal{X}_n$ $(n \geq 1)$

9. Applications

Fact.

(1) Σ is dense in $SU(2)$.

(2) $[O(4,\mathcal{R}) : W^\infty] = 2$ and $O(4,\mathcal{R})$ is dense in $O(4,\mathbb{R})$.

Quantum bit.

A quantum bit is of the form $\lambda|0\rangle + \mu|1\rangle$ for $\lambda, \mu \in \mathbb{C}, |\lambda|^2 + |\mu|^2 = 1$.

Σ is dense in the set of all quantum bits.

Discretization and Approximation.

(1) There is an approximation in $O(4,\mathbb{R})$ using 5 generators of W^∞.

(2) There is an application of Σ as transformation of quantum bits.

Let $G = \Sigma = SU(2, \mathcal{R})$, and put

$$v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathcal{V} = G v_+ \leftrightarrow \Sigma, \ \mathcal{V}_n = \mathcal{X}_n v_+ \leftrightarrow \mathcal{X}_n.$$

Then,

$$\mathcal{V} = \bigcup_{n \geq 0} \mathcal{V}_n \leftrightarrow \Sigma = \bigcup_{n \geq 0} \mathcal{X}_n \subset \mathbb{H}_1,$$

and

$$(G, \mathcal{V}) \text{ is an approximation of } (SU(2), Q\text{-bits}).$$

Furthermore,

$$\sharp\, \mathcal{V}_n = \sharp\, \mathcal{X}_n = 3 \cdot 2^{2n+2} \ \text{ for } \ n \geq 2.$$

In particular, $(SU(2), Q\text{-bits})$ can be discretized.

Note $\mathcal{X}_0 = \Delta_0, \ \mathcal{X}_1 = \Delta_1, \ \mathcal{X}_2 = \Xi \cup \Xi', \ \mathcal{X}_0 \cup \mathcal{X}_1 \cup \mathcal{X}_2 = \Delta \cup \Delta',$

and recall $\sharp\, \mathcal{X}_0 = 8, \ \sharp\, \mathcal{X}_1 = 16, \ \sharp\, \mathcal{X}_2 = 192.$

Happy 70th Birthday, Kolya !

Please be healthy,

please be safe, and

please be peaceful !!