Fields of U-invariants

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Abstract. The general linear group $\mathrm{GL}(n)$ acts on the linear space of matrix m-tuples by the adjoint action. The action of $\mathrm{GL}(n)$ induces the action of the unitriangular subgroup U. We present the system of free generators of the field of U-invariants.

The general linear group $\mathrm{GL}(n)$ over a field K acts on the space $\mathcal{H} = \mathrm{Mat}(n)\oplus\ldots\oplus\mathrm{Mat}(n)$ of matrix m-tuples by the adjoint action $\mathrm{Ad}_g(X_1,\ldots,X_m) = (gX_1g^{-1},\ldots,gX_mg^{-1})$. The action of the group $\mathrm{GL}(n)$ on $\mathcal{H}$ defines the representation

$$\rho(g)f(X_1,\ldots,X_m) = f(g^{-1}X_1g,\ldots,g^{-1}X_mg)$$

of the group $\mathrm{GL}(n)$ in the space of regular functions $K[\mathcal{H}]$. This representation is extended to the action of $\mathrm{GL}(n)$ on the field of rational functions $K(\mathcal{H})$. For a given subgroup $G \subseteq GL(n)$, the problem is to describe the algebra (respectively, the field) of invariants with respect to the action of G on $\mathcal{H}$.

In the case $G = \mathrm{GL}(n)$ (or $G = \mathrm{SL}(n)$), this problem is solved in the framework of the classical theory of invariants in tensors (see [1, 2, 3]). The algebra of $\mathrm{GL}(n)$-invariants is generated by the system of polynomials $\mathrm{Tr}(A_{i_1}\cdots A_{i_p})$, where $1 \le i_1,\ldots,i_p \le m$.

The group $\mathrm{GL}(n)$ contains the subgroup of unitriangular matrices $U = \mathrm{UT}(n)$, which consists of the upper triangular matrices with ones on the diagonal. Since U acts on $\mathcal{H}$ by unipotent transformations, the field $K(\mathcal{H})^U$ is a pure transcendental extension of K [4]. Our goal is to present a system of free generators of $K(\mathcal{H})^U$.

For $m = 2$, we have $\mathcal{H}_2 = \mathrm{Mat}(n)\oplus\mathrm{Mat}(n)$. Let $\{x_{ij}\}_{i,j=1}^n$ and $\{y_{ij}\}_{i,j=1}^n$ be the systems of standard coordinate functions on the first and second components of $\mathcal{H}_2$. Consider two matrices

$$X = (x_{ij})_{i,j=1}^n \qquad \text{and} \qquad Y = (y_{ij})_{i,j=1}^n.$$

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For two positive integers a and b, we denote by $[a,b]$ the subset of integers $a \leq i \leq b$. For any integer $1 \leq i \leq n$, let i' be the symmetric number to i with respect to the center of the segment $[1,n]$. We have $i' = n - i + 1$.

For the pair $i' \leq j$ (i.e. (i,j) lies on or below the anti-diagonal), let $M_{ij}(X)$ be the minor of order i' of the matrix X with the system of rows $[i,n]$ and columns $[1,i'-1] \sqcup \{j\}$.

For the pair $j \leq k$, let $N_{jk}(Y)$ be the minor of order k' of the matrix Y with the system of rows $\{j\} \sqcup [k+1,n]$ and columns $[1,k']$.

Let $i' < k$. This is equivalent to the pair (i,k) lies below the ant-diagonal. We define the polynomial

$$P_{ik}(X,Y) = \sum_{i' \leq j \leq k} M_{ij}(X) N_{jk}(Y). \quad (1)$$

Proposition. The polynomials $\{P_{ik}(X,Y):\ i' < k\}$, and $D_k(X),\ D_k(Y)$, where $1 \leq k \leq n$, are U-invariant.

We denote

$$P_{ik}(X) = P_{ik}(X,X) = \sum_{i' \leq j \leq k} M_{ij}(X) N_{jk}(X). \quad (2)$$

Corollary. The polynomials $\{P_{ik}(X):\ i' < k\}$ are U-invariant.

For each $1 \leq i \leq n$, let $D_k(X)$ stand for the lower left corner minor of order k' of the matrix X. Observe that $D_k(X) = M_{k,k'}(X) = N_{k,k}(X)$.

In the case $m = 1$, we have $\mathcal{H} = \mathrm{Mat}(n)$. The group U acts on $\mathrm{Mat}(n)$ by the adjoint representation.

Theorem 1. The field $K(\mathrm{Mat}(n))^U$ is freely generated over K by the system of polynomials

$$\{P_{ik}(X):\ i' < k\} \sqcup \{D_k(X):\ 1 \leq k \leq n\}.$$

This system of free generators of $K(\mathrm{Mat}(n))^U$ is not unique. For the other approach see [5, 6].

For an arbitrary m and the linear space $\mathcal{H} = \mathrm{Mat}(n) \oplus \ldots \oplus \mathrm{Mat}(n)$ of matrix m-tuples, we define the following systems of polynomials:

$$\mathbb{P}_{1,\ell} = \{P_{ik}(X_1, X_\ell):\ \ 1 \leq i' < k \leq n\} \ \text{ for each } \ 2 \leq \ell \leq m,$$

$$\mathbb{P}_{\ell} = \{P_{ik}(X_\ell):\ \ 1 \leq i' < k \leq n\} \ \text{ for each } \ 1 \leq \ell \leq m,$$

$$\mathbb{D}_{\ell} = \{D_k(X_\ell): 1 \leq k \leq n\} \ \text{ for each } \ 1 \leq \ell \leq m.$$

Theorem 2. The field $K(\mathcal{H})^U$ is freely generated over K by the system of polynomials

$$\left(\bigcup_{\ell=2}^{m} \mathbb{P}_{1,\ell} \right) \cup \left(\bigcup_{\ell=1}^{m} \mathbb{P}_{\ell} \right) \cup \left(\bigcup_{\ell=1}^{m} \mathbb{D}_{\ell} \right).$$

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