Elements with almost simple spectrum in representations of simple algebraic groups

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(joint work with D. Testerman)

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Introduction

Let G be a simple algebraic group defined over an algebraically closed field F of characteristic $p \geq 0$, and let ρ be an irreducible representation of G. Let $g \in G$ be a semisimple element. If all but one eigenvalues of ρ are of multiplicity 1 then we say that the spectrum of $\rho(g)$ is *almost simple*. Recall that the spectrum of $\rho(g)$ is called *simple* if all eigenvalues of $\rho(g)$ are of multiplicity 1.

The term "almost simple spectrum" may hint that the content of these two notions is almost the same. However, this is not true.

To emphisize the difference, observe that $g \in H = GL(V)$ has simple spectrum if and only if $C_H(g)$ is abelian whereas saying that g has almost simple spectrum is equaivalent the derived group of $C_H(g)$ to be simple.

Group representations containing an element with simple spectrum attracted a significant attention in the literature.

Probably, the most recent result was obtained in a paper by N. Katz and Pham Tiep (2021) who determine the triples (G, ϕ, g), where G is a finite quasi-simple group, $g \in G$ and ϕ an irreducible representation of G over the *complex numbers*, such that the spectrum of $\phi(g)$ is simple.

A similar problem for quasi-simple groups G of Lie type and for ϕ to be an irreducible representation of G over a field of *defining characteristic* has been solved in two papers of Suprunenko-Z (1998,2000). Here ϕ extends to a representation of the algebraic group $\mathbf{G}$, which allows to use the representation theory of algebraic groups.

It is obvious that saying that $\phi(\mathbf{G})$ has an element of simple spectrum implies all weights of ϕ to have multiplicity 1. In fact in this case the weight spaces of ϕ are the eigenspaces of that element.

This hints that saying that $\phi(\mathbf{G})$ has an element of almost simple spectrum must imply all but one weight of ϕ to be of multiplicity 1, and the weight spaces are exactly eigenspaces of that element.

The latter part of this is false but the former turns out to be true. Moreover, only the weight 0 can be of multiplicity greater than 1. A proof of this is not entirely trivial, and appeared in a joint paper with Testerman in 2021.

Continuing the comparison with the simple spectrum case, I recall that the irreducible representations of $\mathbf{G}$ whose all weights are of multiplicity one were essentially determined by Seitz (1987) and some details have been refined by I. Suprunenko-Z in 1987. Note that the list contains infinite series of representations.

Turning to the almost simple story, one had to determine the irreducible representations of simple algebraic groups whose all but one weights have multiplicity 1. This is done in a joint work with Testerman (2015). This result adds to Seitz's list only a minor number of new cases.

Note that a maximal torus of an algebraic group contains a dense cyclic subgroup. Therefore, this result is equivalent to that of describing the irreducible representations of $\mathbf{G}$ whose image contains some element of almost simple spectrum.

In fact, in the framework of the project one also wishes to describe, given an irreducible representation ρ, the elements $g \in \mathbf{G}$ such that the spectrum of $\rho(g)$ is almost simple.

Note that one has to ignore the central elements of a group as a scalar matrix has almost simple spectrum.

We first considered the question whether g must be regular. One should at once exclude the natural representations of classical groups, due to obvious counter examples. Say, the spectrum of the diagonal matrix diag$(-1, -1, \ldots, -1, 1) \in \mathbf{G} = SL_n(F)$, $n > 2$ odd, is almost simple, but this element is not regular in $\mathbf{G}$.

Note that a semisimple element g of a simple algebraic group $\mathbf{G}$ is regular if and only if $C_{\mathbf{G}}(g)$ is abelian.

In contrast to this example, we show

Theorem 1 (2021). Let ϕ be an irreducible representation of a simple algebraic group $\mathbf{G}$ and $g \in \mathbf{G}$, where g is not regular. Suppose that the spectrum of $\phi(g)$ is almost simple. Then ϕ is a twist of the natural representation of $\mathbf{G}$.

Here we call a representation ϕ a *twist of* ν if $\phi(x) = \nu(\sigma(x))$ for all $x \in \mathbf{G}$ and for some surjective homomorphism $\sigma : \mathbf{G} \to \mathbf{G}$.

There are a few "quasi-exceptions" in this statement arising from the isomorphisms $D_3 \cong A_3$, $B_2 \cong C_2$ and $A_1 \cong B_1 \cong C_1$.
For instance the natural representation of D_3 is not natural when it is viewed as a representation of A_3, which yields an exception unless we agree to regard A_3 as D_3.

What can be said if g is regular? At the first sight, the above result is best possible as a maximal torus always contains a Zarisky dense cyclic subgroup. However, we have observed that g in question must lie in an interesting subset of the set of regular semisimple elements.

Definition. Let T be a maximal torus of $\mathbf{G}$ and $g \in T$. We say that g is strictly regular if g takes distinct values at the roots of $\mathbf{G}$.

Recall that the roots are weights of T and hence every root α yields a homomorphism $T \to F^{\times}$, so the definition is meaningful.

Let ρ be an irreducible representation with highest weight $\omega \neq 0$, and $p \geq 0$ is the defining characteristic of $\mathbf{G}$.
We prove that if $\rho(g)$ has simple spectrum then g is strictly regular, unless ρ is a twist of an irreducible representation whose highest weight is minuscule, or
$(\mathbf{G}, \omega, p) \in \{(F_4, p^k\omega_4, 2),\ (G_2, p^k\omega_1, p), (G_2, p^k\omega_2, 3)\}$
and some cases for $(\mathbf{G}, p) \in \{(C_n, 2), (G_2, 2), (G_2, 3)\}$
for ρ to be tensor-decomposable.

Moreover, we have a rather similar result under assumption that the spectrum of $\rho(g)$ is almost simple (published online in 2022).

I am not sure that the explicit form of the result must be displayed here.

The next problem we are interested is the following:

Problem 1. Determine a sharp upper bound for the multiplicity of eigenvalues of $\rho(g)$ when the spectrum of $\rho(g)$ is almost simple.

The example with

$$g = \text{diag}(-1, -1, , \ldots, -1, 1) \in SL_n(F)$$

(n odd), shows that the multiplicity of an eigenvalue of g can be equal to $\dim \rho - 1$. This looks disappointed.

However, we have:

Theorem 2. Let $\mathbf{G}$ be a simple algebraic group of rank n and ρ an irreducible representation of $\mathbf{G}$. Let $g \in \mathbf{G}$ be a non-central semisimple element such that the spectrum of $\rho(g)$ is almost simple.
Then the greatest eigenvalue multiplicity of $\rho(g)$ does not exceed $2n$.

Here the bound is given in terms of the rank of $\mathbf{G}$. If, in addition, g is regular then the bound can be improved to $n+1$ in place of $2n$. Note that the dimensions of irreducible representations with all eigenvalue multiplicities equal 1 are unbounded, even for G of type A_2 so Theorem 2 looks essential.

Recall that, with known exceptions, the element g must be strictly regular.

For such g there is a result on maximal eigenvalue multiplicity m of $\rho(g)$ without assuming $\rho(g)$ to be of almost simple spectrum.
It states that $m \leq \dim \rho/n$ for classical groups, where n is the rank of $\mathbf{G}$, and $m \leq \dim \rho/30$ for exceptional groups. (This is essentially due to Seitz (1997)).

This is a very nice result, but stated in terms of both dim ρ and the rank of $\mathbf{G}$.

In general the bound cannot be less than n, the rank of G, as this attains at the adjoint representation. However, we think that the following problem deserves to be consider:

Problem 2. Determine the irreducible representations ρ of G such that the spectrum of $\rho(g)$ is almost simple and some semisimple element has an eigenvalue of multiplicity greater than $n/2$.

At the moment I cannot state any reasonable conjecture.
Note that tensor-decomposable representations are rather special in this context, and our experience is yet insufficient to comment this case.
However, there are examples with $G = SL_{3m}(F)$ and ρ of highest weight $(1+p)\omega_1$ where g is a regular, the spectrum of $\rho(g)$ is almost simple and some eigenvalue multiplicity of $\rho(g)$ is m.

Examples. Let $G = SL_{3m}(F)$ and $H = \mathrm{diag}(H_1, \dots, H_m)$, where $H_1 \cong \cdots \cong H_m \cong SL_3(F)$. Let p be the characteristic of F and let $p_m > p_{m-1} > \cdots > p_1 > p$ be primes. Let $h_i = \mathrm{diag}(d_i^p, d_i^{-1}, d_i^{1-p})$, where d_i is a primitive p_i-root of unity for $i = 1, \dots, m$. Set $g = \mathrm{diag}(h_1, \dots, h_m)$.

Let ρ be an irreducible representation of G with highest weight $(1+p)\omega_1$. Then the spectrum of $\rho(g)$ is almost simple and the eigenvalue 1 multiplicity of $\rho(g)$ equals m.

Note that $\mathrm{diag}(H_1, \dots, H_m)$ means a block-diagonal matrix with diagonal blocks $H_1, \dots, H_m$.

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