Chow-weight homology: a "mixed motivic decomposition of the diagonal"

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Homological motives: short reminder

k = perfect base field of characteristic p ($p \geq 0$); Var $\supset$ SmVar $\supset$ SmPrVar = (smooth, projective; possibly reducible) varieties $/k$.

$e = p$ if $p > 0$; $e = 1$ if $p = 0$. $R = \mathbb{Z}[\frac{1}{e}]$ of $R = \mathbb{Q}$.

SmCor_R : Obj = SmVar; morphisms= R-combinations of 'multi-valued functions'.

$\mathrm{Corr}_{\mathrm{R}}^{\mathrm{rat}}$: Obj = $\mathcal{M}_R(\mathrm{SmPrVar})$, morphisms= SmCor_R-ones/ "homotopy equivalence".

$\mathrm{Chow}_R^{\mathrm{eff}} = \mathrm{Kar}(\mathrm{Corr}_{\mathrm{R}}^{\mathrm{rat}})$ (add retracts of $\mathcal{M}_R(\mathbb{P}^1) = L \bigoplus \mathcal{M}_R(pt)(=: R)$;
$\langle 1 \rangle : M \mapsto M \otimes L =: M\langle 1 \rangle$ (1-effective) gives $\mathrm{Chow}_R^{\mathrm{eff}} \cong \mathrm{Chow}_R^{\mathrm{eff}}\langle 1 \rangle \subset \mathrm{Chow}_R^{\mathrm{eff}}$.

$\mathcal{M}_R(\mathbb{P}^n) = R \oplus R\langle 1 \rangle \oplus \cdots \oplus R\langle n \rangle (= L^{\otimes n})$.

Also, $\mathrm{Chow}_R^{\mathrm{eff}} \subset DM_{\mathrm{gm},R}^{\mathrm{eff}}$.

$DM_{\mathrm{gm},R}^{\mathrm{eff}} = \mathrm{Kar}(K^b(\mathrm{SmCor}_R)/\langle HI, MV \rangle)$; contains $\mathcal{M}_R(X)$ for any $X \in$ Var; a tensor category $\supset \mathrm{Chow}_R^{\mathrm{eff}}$.

$j \geq 0$ $\mathrm{CH}_j(M) = \mathrm{Chow}_R^{\mathrm{eff}}(R\langle j \rangle, M)$ =(R-linear) Chow group of dim j cycles for M.

Pure and "classical" statements

$K \supset k$: a universal domain (algebraically closed & of infinite transcendence degree/ $\mathbb{F}_p$, $/\mathbb{Q}$; e.g., $K = \mathbb{C}$).

Theorem 1. 1. [[Blo80]+...] $\forall r > 0$, $M \in \mathrm{Chow}_{\mathbb{Q}}^{\mathrm{eff}}$: $M \in \mathrm{Chow}_{\mathbb{Q}}^{\mathrm{eff}}\langle r \rangle \iff \mathrm{CH}_j(M_K) = \{0\}\ \forall 0 \le j < r$.
2. If $M = M' \otimes \mathbb{Q}$, $M \in \mathrm{Chow}_{\mathbb{Z}[\frac{1}{e}]}^{\mathrm{eff}}$, (1) $\implies \exists E_{M'} > 0$ $E_{M'}\,\mathrm{CH}_j(M'_{k'}) = \{0\}\ \forall k'/k,\ j < r$.

3. Clear: $k \subset \mathbb{C}$, $X \in \mathrm{SmPrVar}$, $\mathcal{M}_{\mathbb{Q}}(X) = \mathbb{Q} \bigoplus M$ ("decomposition of the diagonal" that corresponds to $id_{\mathcal{M}_{\mathbb{Q}}(X)}$), $q > 0 \implies$ $H^q(X_{\mathbb{C}})_{\mathbb{C}} = F^r H^q(X_{\mathbb{C}})_{\mathbb{C}}$ (r-effective in PHS).

k is finitely generated $\implies H^q_{et}(X_{k^{alg}}, \mathbb{Q}_l)$ is r-effective in $\mathbb{Q}_l[\mathrm{Gal}(k)] - \mathrm{Mod}$ ($\forall \ell \in \mathbb{P} \setminus \{p\}$).

Our "mixed" results

Theorem 2 ([BoS22]). 1. $X \in \mathrm{Var}$,
$\mathrm{CH}_j(X_K, \mathbb{Q}) = \{0\}\ \forall 0 < j < r$,
$\&\, \dim_{\mathbb{Q}} \mathrm{CH}_0(X_K, \mathbb{Q}) < \infty \iff \exists$ distinguished
$N'[-1] \to A \bigoplus N\langle r\rangle \to \mathcal{M}^c_{\mathbb{Q}}(X) \to N'$,
A is a retract of $\sum \mathcal{M}_{\mathbb{Q}}(\mathrm{Spec}\ k_i)$, k_i/k are finite,
$N \in \mathrm{Chow}^{\mathrm{eff}}_{\mathbb{Q}}$, $N' \in DM^{\mathrm{eff}}_{\mathrm{gm},\mathbb{Q} w_{\mathrm{Chow}} \geq 1} =$
the extension-closure of $\cup_{i>0} \mathrm{Chow}^{\mathrm{eff}}_{\mathbb{Q}}[i]$.

2. (1) $\iff \exists E_X > 0 : E_X\, \mathrm{CH}_j(X_{k'}, \mathbb{Z}[\frac{1}{e}]) = \{0\}$
$\forall k'/k$, $0 < j < r$, $\& E_X\, \mathrm{CH}_0(X_{k'}, \mathbb{Z}[\frac{1}{e}]) \cong \mathbb{Z}[\frac{1}{e}]^m$.

3. $Gr_q^{W_D}(H_c^q(X))$ is r-effective $\forall q > 0$
($k \subset \mathbb{C}$ or finite dimensional; H as above).

4. "Standard" conjectures $\implies$ (3) $\iff$ (1).

5. $X = X^1 \times X^2$, $\mathrm{CH}_j(X^i_K, \mathbb{Q}) = \{0\}\ \forall j \leq r_i$,
$i = 1, 2 \implies \mathrm{CH}_j(X, \mathbb{Q}) = \{0\}\ \forall j \leq r_1 + r_2 + 1$.

Theorem 3. 1. $\forall j \geq 0\ \exists!$ homological $\mathrm{CWH}_j : DM^{\mathrm{eff}}_{\mathrm{gm},R} \to R-\mathrm{Mod}$ that kills $\mathrm{Chow}^{\mathrm{eff}}_R[i]$ $\forall i \neq 0+$ restricts to CH_j on $\mathrm{Chow}^{\mathrm{eff}}_R$ $(\subset DM^{\mathrm{eff}}_{\mathrm{gm},R})$.

2. $M \in DM^{\mathrm{eff}}_{\mathrm{gm},R}$; $M \in DM^{\mathrm{eff}}_{\mathrm{gm},R}\langle r\rangle \iff \mathrm{CWH}^i_j(M_{k'}) = \{0\}\ \forall k'/k,\ 0 \leq j < r,\ i \in \mathbb{Z}$.

3. $R = \mathbb{Q} \implies k' = K$ is "sufficient"; $M = M' \otimes \mathbb{Q}$, $M' \in DM^{\mathrm{eff}}_{\mathrm{gm},\mathbb{Z}[\frac{1}{e}]} \implies \exists E_{M'} > 0 : E_{M'}\,\mathrm{CWH}^i_j(M'_{k'}) = \{0\}\ \forall k', j, i$.

4. (2) $\implies H^*(M)$ is r-effective.

5. $R = \mathbb{Q}+$"standard conjectures" $\implies$ (4) $\iff$ (2).

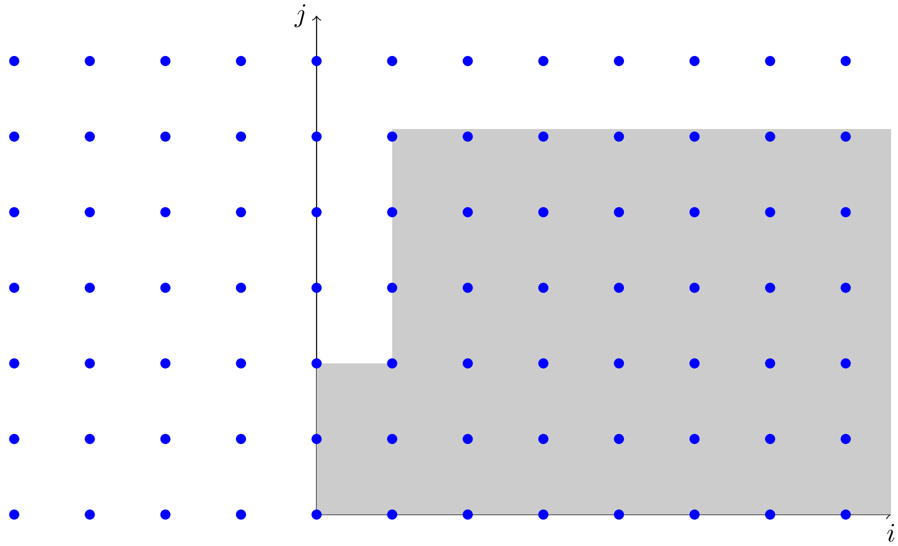
Remark 4. 1. Similar criteria for $\mathrm{CWH}^i_j(M_{k'}) = \{0\}$ $\forall (i,j) \in \mathcal{I}$ if $\mathcal{I} \subset \mathbb{Z} \times [0,+\infty)$, $(i,j) \in \mathcal{I} \implies (i',j') \in \mathcal{I}$ if $(i',j') \in \mathbb{Z} \times [0,+\infty)$, $i' \geq i, j' \leq j$.

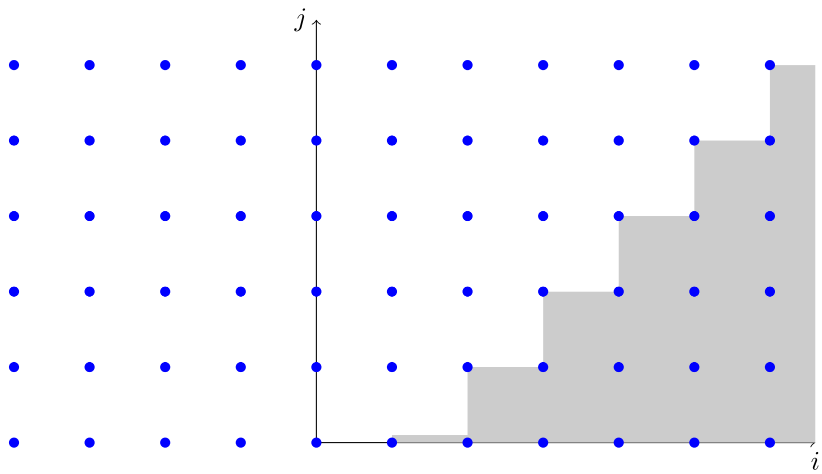
2. In [BoK20] (CWH was extended to DM &): if $(i,j) \in \mathcal{I} \implies (i+c, j+c) \in \mathcal{I}\ \forall c > 0$ then CWH^i_j can be replaced by $DM^{\mathrm{eff}}_{\mathrm{gm},R}(R\langle j\rangle[-i], -)$.

3. CWH can be computed via $t : DM^{\mathrm{eff}}_{\mathrm{gm},R} \to K^b(\mathrm{Chow}^{\mathrm{eff}}_R)$; comes from w_{Chow}.

4. The proofs rely on the Chow weight structure on $DM^{\mathrm{eff}}_{\mathrm{gm},R}/DM^{\mathrm{eff}}_{\mathrm{gm},R}\langle 1\rangle$.

5. Much more was proved; this includes dual results.





Thank you for your attention!

I heartily congratulate Professor Vavilov on his 70th anniversary!

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