Rook placements in G_2 and F_4 and associated coadjoint orbits

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Abstract. Let N be a nilpotent complex Lie group, $\mathfrak{n}$ be its Lie algebra. Given a rook placement D and a map from this rook placement into $\mathbb{C}^{\times}$, one can define the linear form on $\mathfrak{n}$. We are interested in the coadjoint orbits of such forms. Let ξ_1 and ξ_2 be distinct maps from D to $\mathbb{C}^{\times}$, let Ω_{D,ξ_1}, Ω_{D,ξ_2} be the correspondent orbits. M. Ignatev give a conjecture that $\Omega_{D,\xi_1} \neq \Omega_{D,\xi_2}$. I will prove this conjecture for the cases $\Phi = G_2$ and F_4.

Let N be a nilpotent complex Lie group, $\mathfrak{n}$ be its Lie algebra, and $\mathfrak{n}^*$ be its dual space. The group N acts on the Lie algebra $\mathfrak{n}$ via an adjoint representation; the dual representation in the space $\mathfrak{n}^*$ is called coadjoint. According to the orbit method of A.A. Kirillov, the orbits of the coadjoint representation play a key role in the representation theory of the group N [6].

We are interested in the case when N is a unipotent radical of a Borel subgroup B of a simple complex algebraic group G. The complete classification of coadjoint orbits in $\mathfrak{n}^*$ is a wild problem. Therefore, of particular interest is the study of certain important classes or series of orbits. Almost all the orbits that have been studied to date are so-called orbits associated with rook placements in root systems.

Let Φ be the root system of the group G, Φ^+ be the set of positive roots with respect to B, the set $\{e_\alpha, \alpha \in \Phi^+\}$ be the basis of the algebra $\mathfrak{n}$ consisting of root vectors, the set $\{e^*_\alpha, \alpha \in \Phi^+\}$ be the dual basis of space $\mathfrak{n}^*$. A rook placement is a subset $D \subset \Phi^+$ consisting of roots with pairwise non-positive scalar products. For any mapping $\xi\colon D \to \mathbb{C}^{\times}$, we define the linear form $f_{D,\xi} = \sum\limits_{\beta \in D} \xi(\beta) e^*_\beta \in \mathfrak{n}^*$; let $\Omega_{D,\xi} \subset \mathfrak{n}^*$ be its orbit. We will say that the orbit $\Omega_{D,\xi}$ is associated with the rook placement D. For example, in the case $\Phi = A_{n-1}$, all orbits of maximal dimension are associated with the same orthogonal rook placement, called the Kostant cascade.

We call a rook placement D non-singular if the fact that α and β are in D implies that $\alpha - \beta \notin D$. For example, for $\Phi = A_{n-1}$ all rook placements are non-singular.

Orbits associated with orthogonal rook placements (in particular, with Kostant cascades) were studied in detail in [2], [3], [7], [8].

During the study, the following conjecture arose.

Conjecture. *Let D be a non-singular orthogonal rook placement, ξ_1, ξ_2 be different maps from D to $\mathbb{C}^\times$. Then Ω_{D,ξ_1} and Ω_{D,ξ_2} do not coincide.*

For $\Phi = A_{n-1}$ this follows from the results of A.N. Panov [8]. For the remaining classical series B_n, C_n, D_n, the proof of the conjecture essentially reduces to the case A_{n-1}. In [5] M.V. Ignatiev and A.A. Shevchenko proved that the conjecture is true for $\Phi = E_6, E_7$ or E_8 for some (but not all) rook placement. The main result of my report is as follows.

Theorem. *Conjecture is true in cases $\Phi = G_2$ and F_4*

For G_2, this is an easy exercise, while for F_4, a modification of the argument from [6] is required, which, in fact, also reduces the problem to the case A_{n-1}, where stronger results of C. Andre [1] can be applied.

The talk is based on joint work with M.V. Ignatev [4].

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