Width of a congruence subgroup over an arithmetic ring

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Abstract. We give an estimate for the width of the congruence subgroup $\mathrm{SL}(n, \mathcal{O}_S, I)$ in Tits–Vaserstein generators, where $\mathcal{O}_S$ is a localisation of the ring of integers in a number field K. We assume that either K has a real embedding, or the ideal I is prime to the number of roots of unity in K.

Extended abstract

Given a group G with generating set X, the width of G in generators from X is a minimal number N such that any element of G is a product of at most N elements from X. When the width is finite, we say that G admits bounded generation with respect to X.

For special linear group, or more generally for Chevalley groups bounded generation with respect to the elementary generators is known for certain classes of rings. For example this holds for Dedekind domains of arithmetic type, see [1] for the arithmetic case and [3] for the functional case. Moreover, the proofs are effective , i.e they give an estimate for the corresponding width.

In [4], it was proven that for an ideal $I \trianglelefteq \mathcal{O}_S$ the principal congruence subgroup $G(\Phi, \mathcal{O}_S, I)$ of a classical Chevalley group of type Φ with $\mathrm{rk}\,\Phi \leq 2$ over a Dedekind domain $\mathcal{O}_S$ of arithmetic type has finite width in Tits–Vaserstein generators (i.e. generators of type $x_\alpha(\xi)^{x_{-\alpha}(\zeta)}$, where $\alpha \in \Phi$, $\xi \in I$ and $\zeta \in \mathcal{O}_S$), provided the fraction field of $\mathcal{O}_S$ has a real embedding. However, the proof relies on results from [5], which on its turn are not constructive and do not allow to obtain an explicit estimate of the width in question.

The talk is base on the paper [2] by author, where an effective version of the result from [4] is proved for special linear group, i.e. the width of $\mathrm{SL}(n, \mathcal{O}_S, I)$ in Tits–Vaserstein generators is estimated explicitly.

Namely let $\mathcal{O}$ be the ring of integers in an algebraic number field K. Let D be the discriminant of K and $\mathrm{Cl}(K)$ be its class group. Let m be the number of roots of unity in K. For any rational prime p set $e_p = \mathrm{ord}_p(m)$, i.e. $m = \prod_{\{p\colon e_p>0\}} p^{e_p}$. Further for any rational prime p we denote by L_p the extension of K obtained by adjoining a primitive p^{e_p+1}-th root of unity. Now set

$$\mathbb{S}_{\mathrm{bad}} = \{p \in \mathbb{P}\colon p \mid D \text{ and } \gcd([L_p\colon K], |\,\mathrm{Cl}(K)|) > 1\},$$

where $\mathbb{P}$ denotes the set of rational primes. Finally, set

$$\Delta = \max_{\delta_1+\delta_2+\delta_3=|\mathbb{S}_{\mathrm{bad}}|} \left(\sum_{i=1}^{3} \max(1, [\ln(\delta_i+1)/\ln 2]) \right),$$

where maximum is taken over all triples of nonnegative integers δ_1,δ_2,δ_3 with $\delta_1+\delta_2+\delta_3 = |\mathbb{S}_{\mathrm{bad}}|$. The main result that will be discussed in the talk is the following theorem.

Theorem 1. *In the notation above, let S be a multiplicative system in $\mathcal{O}$, let $\mathcal{O}_S = \mathcal{O}[S^{-1}]$, and I be a non zero ideal in $\mathcal{O}_S$. Suppose that either K has a real embedding, or I is prime to m. Let $n \geq 3$. Then the width of* $\mathrm{SL}(n, \mathcal{O}_S, I)$ *in Tits–Vaserstein generators $\{t_{i,j}(\xi)^{t_{j,i}(\zeta)}\colon \xi \in I, \zeta \in \mathcal{O}_S\}$ is at most* $3n(n-1)/2 + 2n + 1632\Delta + 185$.

In the talk we discuss the main ideas of the proof and certain corollaries. We also discuss how the noneffective version of the theorem above (i.e. the proof of bounded generation without the explicit bound) can be easily obtain directly from the result of [1].

References

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