Width of a congruence subgroup over an arithmetic ring

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Width of $\mathrm{SL}(n, \mathcal{O}_S)$

Width of a group

G — group; $X \subseteq G$ — generating set.
$w_X(G)$ = minimal N such that any element of G is a product of at most N elements from X.

Theorem (Carter and Keller, 1983)

Let $\mathcal{O}$ be a ring of integers in an algebraic number field;
$\mathcal{O}_S$ — localisation in some multiplicative set S;
$\Delta = \log_2(\#\{\text{primes dividing the discriminant}\} + 1)$; $n \geqslant 3$;
$G = \mathrm{SL}(n, \mathcal{O}_S)$; $X = \{t_{i,j}(\xi) \colon 1 \leqslant i \neq j \leqslant n,\ \xi \in \mathcal{O}_S\}$.
Then $w_X(G) \leqslant (3n^2 - n)/2 + 68\Delta + 2$.

Relative case

Let $I \trianglelefteq \mathcal{O}_S$ be a proper ideal.

Relative elementary subgroup

$E(n, I) = \langle t_{i,j}(\xi) \colon 1 \leqslant i \neq j \leqslant n,\ \xi \in I \rangle$;

$G = E(n, \mathcal{O}_S, I) = E(n, I)^{E(n, \mathcal{O}_S)}$;

Tits–Vaserstein generators

$X = \{t_{i,j}(\xi)^{t_{j,i}(\zeta)} \colon 1 \leqslant i \neq j \leqslant n,\ \xi \in I,\ \zeta \in \mathcal{O}_S\}$.

Question

$w_X(G) \leqslant ?$

Length of $t_{i,j}(\xi)^g$.

Length of an element

$g \in G$;
$l_X(g) =$ minimal N such that g is a product of at most N elements from X.

Theorem (Buryakov and Vavilov, 2022)

Let R be an arbitrary ring; $I \trianglelefteq R$ be an ideal; $\xi \in I$; $g \in \mathrm{GL}(n, R)$;
$X = \{t_{i,j}(\xi)^{t_{j,i}(\zeta)} \colon 1 \leqslant i \neq j \leqslant n, \xi \in I, \zeta \in R\}$;
Then $l_X(t_{i,j}(\xi)^g) \leqslant n\left(\frac{3}{2}n^2 - \frac{3}{2}n - 1\right)$.

Remark

There is a simple but not effective proof that $l_X(t_{i,j}(\xi)^g) < \infty$ by use of generic element.

Simple but not effective proof that $w_X(E(n, \mathcal{O}_S, I)) < \infty$

$|\mathcal{O}_S/I| = r < \infty$. Let $\zeta_1, \ldots, \zeta_r$ be representatives of all the classes.
Let $g \in E(n, \mathcal{O}_S, I) \leqslant E(n, \mathcal{O}_S)$.
By result of Carter and Keller

$$g = \prod_{u=1}^{N} t_{i_u,j_u}(\zeta_{k_u} + \xi_u) = \prod_{u=1}^{N} t_{i_u,j_u}(\zeta_{k_u}) t_{i_u,j_u}(\xi_u),$$

where $\xi_u \in I$ and N does not depend on g. Then

$$g = \underbrace{\prod_{u=1}^{N} t_{i_u,j_u}(\xi_u)^{h_u}}_{\substack{\text{Length bounded by} \\ \text{previous slide}}} \cdot \underbrace{\prod_{u=1}^{N} t_{i_u,j_u}(\zeta_{k_u})}_{\substack{\text{Belongs to a finite} \\ \text{subset of } E(n,\mathcal{O}_S,I)}},$$

where $h_u = \prod_{v=1}^{u} t_{i_u,j_u}(-\zeta_{k_v})$.

Effective version

Let $\mathcal{O}$ be the ring of integers in an algebraic number field K.
D — the discriminant of K; $\mathrm{Cl}(K)$ — the class group; m— the number of roots of unity; I — proper ideal; S — multiplicative system.
For any $p \in \mathbb{P}$ set $e_p = \mathrm{ord}_p(m)$, i.e. $m = \prod_{\{p\colon e_p>0\}} p^{e_p}$; $L_p = K[\sqrt[p^{e_p+1}]{1}]$.

$$\mathbb{S}_{\mathrm{bad}} = \{p \in \mathbb{P}\colon p \mid D \text{ and } \gcd([L_p\colon K], |\mathrm{Cl}(K)|) > 1\},$$

$$\Delta = \max_{\delta_1+\delta_2+\delta_3=|\mathbb{S}_{\mathrm{bad}}|} \left(\sum_{i=1}^{3} \max(1, [\ln(\delta_i+1)/\ln 2]) \right),$$

Theorem (Gvozdevsky, 2022)

Suppose that either K has a real embedding, or I is prime to m. Let $n \geqslant 3$. Then

$$w_X(\mathrm{SL}(n, \mathcal{O}_S, I)) \leqslant 3n(n-1)/2 + 2n + 1632\Delta + 185.$$

Outline of the proof: reduction to $\mathrm{SL}(2, \mathcal{O}_S, I)$.

$$\mathrm{SL}(2, R) \hookrightarrow \mathrm{SL}(n, R) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & & & \\ c & d & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Theorem (Sinchuk and Smolensky, 2018)

Let $I \trianglelefteq R$. Suppose that $\mathrm{sr}(I) \leqslant 2$. Then any element of $\mathrm{SL}(n, R, I)$ can be decomposed into a product of one element of $\mathrm{SL}(2, R, I)$ and at most $3n(n-1)/2 + 2n - 7$ Tits–Vaserstein generators.

Fact

$\mathrm{KrullDim}(R) \leqslant 1 \Rightarrow \mathrm{sr}(I) \leqslant 2$.

Outline of the proof: double of a ring

Definition

$$\widetilde{R} = \{(a, b) \in R \times R \colon a \equiv b \mod I\}.$$

Lemma

Let $g = (g_{i,j}) \in \mathrm{SL}(n, R, I)$. Set $\widetilde{g} = (g_{i,j}, \delta_{i,j}) \in \mathrm{SL}(n, \widetilde{R})$.
If $\widetilde{g}$ is a product of N elementary transvections in $\mathrm{SL}(n, \widetilde{R})$, then g is a product of N elements of type $t_{i,j}(\xi)^g$ in $\mathrm{SL}(n, R)$, where $\xi \in I$.

Proof

$$\widetilde{g} = \prod_{u=1}^{N} t_{i_u,j_u}(\zeta_u + \xi_u, \zeta_u), \quad \prod_{u=1}^{N} t_{i_u,j_u}(\zeta_u) = e,$$

$$g = \prod_{u=1}^{N} t_{i_u,j_u}(\zeta_u + \xi_u) = \prod_{u=1}^{N} t_{i_u,j_u}(\xi_u)^{h_u} \cdot \prod_{u=1}^{N} t_{i_u,j_u}(\zeta_u).$$

Outline of the proof

Theorem

Suppose that either K has a real embedding, or I is prime to m. Then any element of $\mathrm{SL}(2, \widetilde{O_S})$ is a product of at most $68\Delta + 8$ transvections in $\mathrm{SL}(3, \widetilde{O_S})$.

The proof is similar to the one for the absolute case, but trickier.

Hence

Any element of $\mathrm{SL}(2, O_S, I)$ is a product of at most $24(68\Delta + 8)$ Tits–Vaserstein generators in $\mathrm{SL}(3, O_S, I)$.

Hence

Any element of $\mathrm{SL}(n, O_S, I)$ is a product of at most $24(68\Delta + 8) + 3n(n-1)/2 + 2n - 7$ Tits–Vaserstein generators.

Thank you for your attention.