Groups with commutator relations of type A

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Abstract. The general linear group $\mathrm{GL}(n, A)$ over an associative unital ring A contains elementary transvections $t_\alpha(p)$ for α from the root system of type A_{n-1} and $p \in A$. These elements satisfy $t_\alpha(p + q) = t_\alpha(p)\, t_\alpha(q)$ and the Chevalley commutator formula. We show a partial converse to this fact: suppose that a group G has a family of subgroups parametrized by the root system of type A_{n-1} and these subgroups satisfy the Chevalley commutator formula. Then under an additional natural assumption there is a unique homomorphism from the Steinberg group of a generalized matrix ring to G mapping the root subgroups to the distinguished subgroups of G.

Let A be a unital associative ring and $n \geq 4$. The group $\mathrm{GL}(n, A)$ contains the elementary transvections $t_{ij}(p) = 1 + pe_{ij}$ for $i \neq j$ and $p \in A$ satisfying the Steinberg relations

$$\begin{aligned} t_{ij}(p + q) &= t_{ij}(p)\, t_{ij}(q); \\ [t_{ij}(p), t_{jk}(q)] &= t_{ik}(pq) \text{ for } i \neq k; \\ [t_{ij}(p), t_{kl}(q)] &= 1 \text{ for } j \neq k \text{ and } i \neq l. \end{aligned}$$

The unstable Steinberg group $\mathrm{St}(n, A)$ is generated by the elements $x_{ij}(p)$ for $1 \leq i \neq j \leq n$ and $p \in A$ satisfying the Steinberg relations, so there is a canonical homomorphism

$$\mathrm{St}(n, A) \to \mathrm{GL}(n, A), x_{ij}(p) \mapsto t_{ij}(p).$$

Now suppose that we have a group G with subgroups G_{ij} for $1 \leq i < j \leq n$ such that

$$\begin{aligned} [G_{ij}, G_{jk}] &\leq G_{ik} \text{ for } i \neq k; \\ [G_{ij}, G_{kl}] &= 1 \text{ for } j \neq k \text{ and } i \neq l. \end{aligned}$$

In other words, G has A_{n-1}-commutator relations in the sense of [2, chapter I]. If G in addition has Weyl elements n_{ij} permuting the subgroups G_{ij}, then under natural assumption there are a unital associative ring A and a homomorphism $f\colon \mathrm{St}(n, A) \to G$ such that f maps $A \cong x_{ij}(A)$ isomorphically to G_{ij}, see [3].

We are interested in the more general case where G does not contain Weyl elements. For example, let R be a generalized matrix ring, i.e. a non-unital associative ring with a Peirce decomposition

$$\begin{aligned} R &= \bigoplus_{1 \leq i,j \leq n} R_{ij}; \\ R_{ij} R_{jk} &\leq R_{ik}; \\ R_{ij} R_{kl} &= 0 \text{ for } j \neq k. \end{aligned}$$

An element $x \in R$ is called quasi-invertible if it is invertible with respect to the monoid operation $x \circ y = xy + x + y$. Let $G = R^\circ$ be the group of quasi-invertible elements of R. It contains the subgroups $G_{ij} = R_{ij}$ satisfying the commutator relations. Clearly, there are no Weyl elements in G in general, e.g. for

$$\begin{aligned} R &= \mathrm{M}(5, \mathbb{Z}); \\ R_{ij} &= \mathbb{Z} e_{ij} \text{ for } i, j < 4; \\ R_{i4} &= \mathbb{Z} e_{i4} \oplus \mathbb{Z} e_{i5} \text{ for } i < 4; \\ R_{4j} &= \mathbb{Z} e_{4j} \oplus \mathbb{Z} e_{5j} \text{ for } j < 4; \\ R_{44} &= \mathbb{Z} e_{44} \oplus \mathbb{Z} e_{45} \oplus \mathbb{Z} e_{54} \oplus \mathbb{Z} e_{55} \end{aligned}$$

the subgroups G_{ij} are not isomorphic.

Our main result [1] is the following: let G be a group with subgroups G_{ij} satisfying the commutator conditions and an additional natural assumption (a stronger version of the condition $[G_{ij}, G_{jk}] = G_{ik}$ for $i \neq k$). Then there is a generalized matrix ring R and a homomorphism from its Steinberg group to G mapping the root subgroups isomorphically to G_{ij}, so $R_{ij} \cong G_{ij}$.

References

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