On the structure of some unitary Nil K_1-groups

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Abstract. We introduce several Nil-subgroups of the unitary Bass' nilpotent K_1-group of a unitary ring and study structure of these Nil-groups. Obtaining properties of the Nil-groups are unitary analogues well-known properties of the Bass' nilpotent K_1-group of a ring in algebraic $K-$theory.

Introduction

In the paper, we follow the standard setting and notation of unitary (algebraic) $K-$theory [1, 2]. First we recall basic definitions and results used in the sequel.

Let (R, λ, Λ) be a unitary ring, alias Bak's form ring, where R is an associative ring with 1, equipped with an involution $x \to \bar{x}$. Further, let λ be a central element of R such that $\lambda \cdot \bar{\lambda} = 1$, and let Λ be an additive subgroup of R such that $\{x - \lambda\bar{x}, x \in R\} \leq \Lambda \leq \{x \in R : x = -\lambda\bar{x}\}$. We note that $(R, \bar{\lambda}, \overline{\Lambda})$, where $\overline{\Lambda} = \{\bar{x}, x \in \Lambda\}$, is a unitary ring also. Let us extend the involution to the matrix ring $M_r(R)$ by setting $(a_{ij})^* = (\overline{a_{ji}})$.

Definition 1. A matrix $a (\in M_r(R))$ is said to be $\Lambda-$hermitian if the a is a $(-\lambda)-$hermitian, i.e., $a = -\lambda a^*$, and all diagonal entries of the a belong to Λ. It is obvious, a matrix a is $\Lambda-$hermitian if and only if a^* is $\bar{\Lambda}-$hermitian.

In the paper we write matrices in a block form. Namely, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2r}(R)$ means that the components $a, b, c, d \in M_r(R)$. For a natural number r, we set $I_r^\lambda = \begin{pmatrix} 0 & e_r \\ \lambda e_r & 0 \end{pmatrix}$, where e_r is the identity matrix of degree r. We note, that I_r^λ is an invertible $\Lambda-$Hermitian matrix, and $(I_r^\lambda)^{-1} = (I_r^\lambda)^*$.

Definition 2. A matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2r}(R)$ is said to be unitary if $\alpha^* I_r^\lambda \alpha = I_r^\lambda$; and the α to be $\Lambda-$unitary if moreover the diagonal entries of the matrices a^*c and b^*d are contained in Λ.

The set $U_{2r}^\lambda(R, \Lambda)$ of all $\Lambda-$unitary matrices of degree $2r$ forms a group; it is called the (hyperbolic) $\Lambda-$unitary group.

Denote by $EU^{\lambda}_{2r}(R,\Lambda)$ the subgroup of $U^{\lambda}_{2r}(R,\Lambda)$, generated by the matrices $H(a)=\begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1}\end{pmatrix}=diag(a,(a^*)^{-1})$ (hyperbolic matrix), $\begin{pmatrix} e_r & b \\ 0 & e_r\end{pmatrix}$, $\begin{pmatrix} e_r & 0 \\ c & e_r\end{pmatrix}$, where $a\in E_r(R)$, b is $\overline{\Lambda}-$hermitian, and c is $\Lambda-$hermitian. The group $EU^{\lambda}_{2r}(R,\Lambda)$ is called the elementary (hyperbolic) $\Lambda-$unitary group.

Let us define an embedding $U^{\lambda}_{2r}(R,\Lambda)\to U^{\lambda}_{2r+2}(R,\Lambda)$:

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\to\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1\end{pmatrix},$$

and set $U^{\lambda}(R,\Lambda)=\cup U^{\lambda}_{2r}(R,\Lambda)$, and $EU^{\lambda}(R,\Lambda)=\cup EU^{\lambda}_{2r}(R,\Lambda)$.

In view of the unitary analog of the Whitehead lemma ([2], Chap.2, Proposition 3.7), the group $EU^{\lambda}(R,\Lambda)$ coincides with the commutator subgroup of the group $U^{\lambda}(R,\Lambda)$. In particular, the (abelian) group $K_1U^{\lambda}(R,\Lambda)=U^{\lambda}(R,\Lambda)/EU^{\lambda}(R,\Lambda)$ is well defined. The class of a matrix $\alpha\in U^{\lambda}(R,\Lambda)$ in the group $K_1U^{\lambda}(R,\Lambda)$ is denoted by $[\alpha]$. As a result, we obtain a unitary K_1-functor K_1U acting from the category of unitary rings to the category of abelian groups.

We extend the involution to the polynomial ring $R[X]$ by setting $\bar{X}=X$. Then $(R[X],\Lambda[X],\lambda)$ is a unitary ring also.

Definition 3. Denote by $NK_1U^{\lambda}(R,\Lambda)$ the kernel of the (splitting) group epimorphism $K_1U^{\lambda}(R[X],\Lambda[X])\to K_1U^{\lambda}(R,\Lambda)$ induced by unitary surjection of unitary rings $(R[X],\Lambda[X])\to(R,\Lambda):X\to 0$. We say that it is the unitary Bass' nilpotent K_1-group of the unitary ring R.

The following statement from [4] should be considered as a unitary analog of the Higman linearization trick, while, in contrast to the linear case of algebraic $K-$theory, only the upper half of the $\Lambda[X]-$unitary matrix, representing an element of the group $NK_1U^{\lambda}(R,\Lambda)$, linearized. In fact, one can carry out a similar linearization of any of the halves of the representing matrix.

Proposition 1 ([4], Theorem 1). Every element of the group $NK_1U^{\lambda}(R,\Lambda)$ has a representative of the form

$$\begin{pmatrix} e_r-aX & bX \\ -cX^n & e_r+a^*X+\cdots+(a^*)^nX^n\end{pmatrix}(\in U^{\lambda}_{2r}(R[X],\Lambda[X]))$$

for some positive integers r and n, where $a,b,c(\in M_r(R))$ satisfy the following conditions:

1) the matrices b and ab are $\bar{\Lambda}-$hermitian and also $ab=ba^*$;

2) the matrices c and ca are $\Lambda-$hermitian and also $ca=a^*c$;

3) $bc=a^{n+1}$ and $cb=(a^*)^{n+1}$.

It is not difficult to show, that the matrix in Proposition 1 is $\Lambda[X]-$unitary if and only if the conditions 1)-3) are satisfied ([4]).

It is well known ([3], Chap.12, Corollary 5.3), that for an arbitrary associative ring R with 1, every element of the Bass' nilpotent K_1-group $NK_1(R)$ has a unipotent representative of the form $e_r - aX$ for some positive integer r, where $a(\in M_r(R))$ is a nilpotent matrix. A similar result for the unitary Bass' nilpotent group $NK_1U^\lambda(R,\Lambda)$ does not generally hold. More exactly, the following statement is fairly.

Proposition 2 ([4], Theorem 2). Let matrices $a, b, c \in M_r(R)$ satisfy the conditions 1)-3) of Proposition 1 for some positive integers r and n. Then, in the notations of Proposition 1, the following statements hold:

1) for $n = 1$, matrix

$$\begin{pmatrix} e_r - aX & bX \\ -cX & e_r + a^*X \end{pmatrix} = e_{2r} - \begin{pmatrix} a & -b \\ c & -a^* \end{pmatrix} X (\in U^\lambda_{2r}(R[X], \Lambda[X]))$$

is a unipotent and the matrix $\begin{pmatrix} a & -b \\ c & -a^* \end{pmatrix}$ is a nilpotent matrix of nilpotency degree 2;

2) for $n \geq 2$, matrix

$$\begin{pmatrix} e_r - aX & bX \\ -cX^n & e_r + a^*X + \cdots + (a^*)^n X^n \end{pmatrix} (\in U^\lambda_{2r}(R[X], \Lambda[X]))$$

is a unipotent if and only if the matrix $e_r - aX$ is unipotent; in this case, the class of this matrix in the group $NK_1U^\lambda(R,\Lambda)$ coincides with the class of the hyperbolic matrix $H(e_r - aX)$.

Now we formulate the main result of the paper.

Theorem. Let α be a nonzero matrix in $M_{2r}(R)$. If $e_{2r} - \alpha X^m \in U^\lambda_{2r}(R,\Lambda)$ for some natural number m then α is a nilpotent matrix of nilpotency degree 2; in this case, the matrix α has the form $\begin{pmatrix} a & -b \\ c & -a^* \end{pmatrix}$, where matrices $a, b, c (\in M_r(R))$ satisfy the conditions 1)-3) of Proposition 1.

We introduce several Nil-groups and represent some properties of these Nil-groups, which are unitary analogues well-known properties of the Bass' nilpotent K_1-group of a ring in algebraic $K-$theory [5].

We denote by $UnipK_1U^\lambda(R,\Lambda)$ the subgroup of $NK_1U^\lambda(R,\Lambda)$ is generated by all elements of the following two types:

1) $[e_{2r} - \alpha X^m]$ for some natural numbers r, m, where $\alpha(\in M_{2r}(R))$ is a nilpotent matrix of nilpotency degree 2;

2) $[H(e_r - aX)]$ for some natural number r, where $a(\in M_r(R))$ is a nilpotent matrix.

Moreover we denote by $Unip_1K_1U^\lambda(R,\Lambda)$ (respectively, $Unip_2K_1U^\lambda(R,\Lambda)$) the subgroup of $UnipK_1U^\lambda(R,\Lambda)$ is generated by all elements of the first type (respectively, second type).

Corollary 1. Let n be a positive integer such that $n = n \cdot 1 = 0$ in the ring R, where 1 is the identity element of R. Then the group $Unip_1K_1U^\lambda(R,\Lambda)$ is of a $n-$torsion group.

Corollary 2. If p is a prime number such that $p^k = 0$ in R for some natural number k, then $UnipK_1U^\lambda(R,\Lambda)$ is a $p-$group.

Corollary 3. If n is an invertible element of R, then the group $Unip_1K_1U^\lambda(R,\Lambda)$ is a uniquely divisible by n.

In unitary (algebraic) $K-$theory for any unitary ring R there exist two standard group homomorphisms: the hyperbolic homomorphism $H : K_1(R) \longrightarrow K_1U^\lambda(R,\Lambda) : [a] \longrightarrow [H(a)]$ and the forgetful homomorphism $F : K_1U^\lambda(R,\Lambda) \longrightarrow K_1(R) : \alpha mod EU^\lambda(R,\Lambda) \longrightarrow \alpha mod E(R)$.

Corollary 4. Under the condition of the Corollary 3, if moreover either the hyperbolic homomorphism H or the forgetful homomorphism F is a monomorphism, then the group $Unip_2K_1U^\lambda(R,\Lambda)$ is a uniquely divisible by n.

Corollary 5. If R is a $\mathbf{Q}-$algebra, where $\mathbf{Q}$ denotes the field of rational numbers, then $Unip_1K_1U^\lambda(R,\Lambda)$ is a $\mathbf{Q}-$vector space. Moreover if either the homomorphism H or the homomorphism F is a monomorphism, then the group $Unip_2K_1U^\lambda(R,\Lambda)$ is a $\mathbf{Q}-$vector space also.

Corollary 6. Under the condition of the Corollary 5, the groups $Unip_1K_1U^\lambda(R,\Lambda)$ and $Unip_2K_1U^\lambda(R,\Lambda))$ are a divisible groups. In particular, these groups are a direct summand both the group $NK_1U^\lambda(R,\Lambda)$ and the group $UnipK_1U^\lambda(R,\Lambda)$.

Conclusion

In Theorem, the main result of the paper, a system of unitary unipotent matrices is found. Using the system of matrices obtained in Theorem, we introduce several Nil-subgroups of the unitary Bass' nilpotent K_1-group of a unitary ring and represent some properties of these Nil-groups. These properties are unitary analogues well-known properties of the Bass' nilpotent K_1-group of a ring in algebraic $K-$theory [5].

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