ON THE STRUCTURE OF SOME UNITARY NIL K_1–GROUPS.

Viacheslav Kopeiko

Kalmyk State University, Elista, Russia
kopeiko52@mail.ru

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$\Lambda-$Hermitian and $\Lambda-$unitary matrices

Let $R = (R, \lambda, \Lambda)$ be a unitary ring, where R is an associative ring with 1, equipped with an involution $x \to \bar{x}$, $\lambda \in C(R)$ such that $\lambda \cdot \bar{\lambda} = 1$, $\Lambda \leq (R, +) : \bar{x}\Lambda x \subseteq \Lambda \ \forall x \in R$ and $\Lambda_{min} \leq \Lambda \leq \Lambda_{max}$, where $\Lambda_{min} = \{x - \lambda\bar{x}, x \in R\}$, and $\Lambda_{max} = \{x \in R : x = -\lambda\bar{x}\}$.
$(R, \bar{\lambda}, \overline{\Lambda})$, where $\overline{\Lambda} = \{\bar{x}, x \in \Lambda\}$, is a unitary ring also.
Extend the involution to the matrix ring $M_r(R) : (a_{ij})^* = (\overline{a_{ji}})$.
Definition 1. A matrix $a = (a_{ij}) \in M_r(R)$ is said to be $\Lambda-$Hermitian, if $a = -\lambda a^*$ and $a_{ii} \in \Lambda$.
We set $I_r^\lambda = \begin{pmatrix} 0 & e_r \\ \lambda e_r & 0 \end{pmatrix}$, where e_r is the identity matrix of degree r.
Definition 2. Let $a, b, c, d \in M_r(R)$. A matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is said to be $\Lambda-$unitary if $\alpha^* I_r^\lambda \alpha = I_r^\lambda$, and the diagonal entries of the matrices a^*c and b^*d are contained in Λ.
The set $U_{2r}^\lambda(R, \Lambda)$ of all $\Lambda-$unitary matrices of degree $2r$ forms a group; it is called the (hyperbolic) $\Lambda-$unitary group.

Unitary K_1–functor

Denote by $EU_{2r}^{\lambda}(R,\Lambda)$ the subgroup of $U_{2r}^{\lambda}(R,\Lambda)$, generated by all the matrices of the form $H(a) = diag(a,(a^*)^{-1})$ (hyperbolic matrix), $T_{12}(b) = \begin{pmatrix} e & b \\ 0 & e \end{pmatrix}$, $T_{21}(c) = \begin{pmatrix} e & 0 \\ c & e \end{pmatrix}$, where $a \in E_r(R)$, b is $\overline{\Lambda}$–Hermitian, c is Λ–Hermitian. The group $EU_{2r}^{\lambda}(R,\Lambda)$ is called the elementary Λ–unitary group.

We set $U^{\lambda}(R,\Lambda) = \cup U_{2r}^{\lambda}(R,\Lambda)$, and $EU^{\lambda}(R,\Lambda) = \cup EU_{2r}^{\lambda}(R,\Lambda)$.

By the unitary analog of the Whitehead lemma, $EU^{\lambda}(R,\Lambda)$ coincides with the commutator subgroup of the group $U^{\lambda}(R,\Lambda)$. In particular, the (abelian) group $K_1U^{\lambda}(R,\Lambda) = U^{\lambda}(R,\Lambda)/EU^{\lambda}(R,\Lambda)$ is well defined. The class of a matrix $\alpha \in U^{\lambda}(R,\Lambda)$ in the group $K_1U^{\lambda}(R,\Lambda)$ is denoted by $[\alpha]$. As a result, we obtain a unitary K_1–functor K_1U acting from the category of unitary rings to the category of abelian groups.

Unitary Bass' nilpotent K_1−group

Definition 3. The kernel of the (splitting) group epimorphism $K_1 U^\lambda(R[X], \Lambda[X]) \to K_1 U^\lambda(R, \Lambda)$, induced by the mapping $(R[X], \Lambda[X]) \to (R, \Lambda) : X \to 0$, we denoted by $NK_1 U^\lambda(R, \Lambda)$ and called unitary Bass' nilpotent K_1−group.

Proposition 1 ([3], Theorem 1). Every element of the group $NK_1 U^\lambda(R, \Lambda)$ has a representative of the form

$$\begin{pmatrix} e_r - aX & bX \\ -cX^n & e_r + a^* X + \cdots + (a^*)^n X^n \end{pmatrix} (\in U^\lambda_{2r}(R[X], \Lambda[X]))$$

where $a, b, c (\in M_r(R))$ satisfy the following conditions:

1) the matrices b and ab are $\bar{\Lambda}$−hermitian and also $ab = ba^*$;

2) the matrices c and ca are Λ−hermitian and also $ca = a^* c$;

3) $bc = a^{n+1}$ and $cb = (a^*)^{n+1}$ for some natural numbers r and n.

The conditions 1)-3) are necessary and sufficient conditions for the matrix in Proposition 1 to be $\Lambda[X]$−unitary.

The statement is a unitary analog of the Higman linearization trick.

Unipotent representatives of the group $NK_1U^\lambda(R,\Lambda)$

Proposition 2 ([4], Theorem 2). Let be matrices $a, b, c \in M_r(R)$ satisfy the conditions 1)-3) of Proposition 1 for some positive integers r and n. Then, in the notations of Proposition 1, the following statements hold:
1) for $n = 1$, matrix

$$\begin{pmatrix} e_r - aX & bX \\ -cX & e_r + a^*X \end{pmatrix} = e_{2r} - \begin{pmatrix} a & -b \\ c & -a^* \end{pmatrix} X (\in U^\lambda_{2r}(R[X], \Lambda[X]))$$

is a unipotent, where $\begin{pmatrix} a & -b \\ c & -a^* \end{pmatrix}$ is a nilpotent matrix of nilpotency degree 2;
2) for $n \geq 2$, matrix

$$\begin{pmatrix} e_r - aX & bX \\ -cX^n & e_r + a^*X + \cdots + (a^*)^n X^n \end{pmatrix} (\in U^\lambda_{2r}(R[X], \Lambda[X]))$$

is a unipotent if and only if the matrix $e_r - aX$ is unipotent; in this case, the class of this matrix in the group $NK_1U^\lambda(R,\Lambda)$ coincides with the class of the hyperbolic matrix $H(e_r - aX)$.

Unitary Nil K_1-groups

Theorem. Let α be a nonzero matrix in $M_{2r}(R)$. If $e_{2r} - \alpha X^m \in U_{2r}^{\lambda}(R, \Lambda)$ for some natural number m then α is a nilpotent matrix of nilpotency degree 2; in this case, the matrix α has the form $\begin{pmatrix} a & -b \\ c & -a^* \end{pmatrix}$, where matrices $a, b, c (\in M_r(R))$ satisfy the conditions 1)-3) of Proposition 1.

We denote by $UnipK_1U^{\lambda}(R, \Lambda)$ the subgroup of $NK_1U^{\lambda}(R, \Lambda)$ generated by all elements of the following two types:

1) $[e_{2r} - \alpha X^m]$ for some natural numbers r, m, where $\alpha(\in M_{2r}(R))$ is a nilpotent matrix of nilpotency degree 2;

2) $[H(e_r - aX)]$ for some natural number r, where $a(\in M_r(R))$ is a nilpotent matrix.

Moreover we denote by $Unip_1K_1U^{\lambda}(R, \Lambda)$ (respectively, $Unip_2K_1U^{\lambda}(R, \Lambda)$) the subgroup of $UnipK_1U^{\lambda}(R, \Lambda)$ generated by all elements of the first type (respectively, second type).

Properties of the unitary Nil K_1-groups, I

Corollary 1. Let n be a positive integer such that $n = n \cdot 1 = 0$ in the ring R, where 1 is the identity element of R. Then the group $Unip_1 K_1 U^{\lambda}(R, \Lambda)$ is of a $n-$torsion group.
Corollary 2. If p is a prime number such that $p^k = 0$ in R for some natural number k, then $UnipK_1 U^{\lambda}(R, \Lambda)$ is a $p-$group.
Corollary 3. If n is an invertible element of R, then the group $Unip_1 K_1 U^{\lambda}(R, \Lambda)$ is a uniquely divisible by n.
In unitary (algebraic) $K-$theory for every unitary ring R there exist two standard group homomorphisms: the hyperbolic homomorphism $H : K_1(R) \longrightarrow K_1 U^{\lambda}(R, \Lambda) : [a] \longrightarrow [H(a)]$ and the forgetful homo $F : K_1 U^{\lambda}(R, \Lambda) \longrightarrow K_1(R) : \alpha modEU^{\lambda}(R, \Lambda) \longrightarrow \alpha modE(R)$.
Corollary 4. Under the condition of the Corollary 3, the group $Unip_2 K_1 U^{\lambda}(R, \Lambda)$ is a divisible by n. Moreover if either the homomorphism H or the homomorphism F is a monomorphism, then the group $Unip_2 K_1 U^{\lambda}(R, \Lambda)$ is a uniquely divisible by n.

Properties of the unitary Nil K_1-groups, II

Corollary 5. If R is a $\mathbf{Q}-$algebra, where $\mathbf{Q}$ denotes the field of rational numbers, then $Unip_1 K_1 U^\lambda(R,\Lambda)$ is a $\mathbf{Q}-$vector space. Moreover if either the homomorphism H or the homomorphism F is a monomorphism, then the group $Unip_2 K_1 U^\lambda(R,\Lambda)$ is a $\mathbf{Q}-$vector space also.
Corollary 6. Under the condition of the Corollary 5, the groups $Unip_1 K_1 U^\lambda(R,\Lambda)$ and $Unip_2 K_1 U^\lambda(R,\Lambda)$) are a divisible groups. In particular, these groups are a direct summand both the group $NK_1 U^\lambda(R,\Lambda)$ and the group $UnipK_1 U^\lambda(R,\Lambda)$.
These properties are unitary analogues well-known properties of the Bass' nilpotent K_1-group of a ring in algebraic $K-$theory [5].

Example, I

In conclusion I represent a linearization form of the unitary unipotent matrix $e_{2r} - \begin{pmatrix} a & -b \\ c & -a^* \end{pmatrix} X^2$, where the matrices b and ab are $\bar{\Lambda}-$Hermitian, the matrices c and ca are $\Lambda-$Hermitian, and $bc = a^2$. It is the following matrix

$$\Gamma = \begin{pmatrix} e_{8r} - \alpha X & \beta X \\ -\gamma X^7 & e_{8r} + \alpha^* X + \cdots + (\alpha^*)^7 X^7 \end{pmatrix},$$

$$\alpha = \begin{pmatrix} 0 & -a & e & 0 & e & 0 & 0 & 0 \\ -e & 0 & 0 & e & 0 & e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Example, II

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & -b & -ab & 0 \\ 0 & 0 & 0 & 0 & -b & 0 & 0 & -b \\ 0 & 0 & 0 & -b & 0 & b & 0 & 0 \\ 0 & 0 & -b & 0 & b & 0 & 0 & 0 \\ 0 & -b & -ab & 0 & 0 & 0 & 0 & 0 \\ -b & 0 & 0 & -b & 0 & 0 & 0 & 0 \end{pmatrix},$$

Example, III

$$\gamma = \begin{pmatrix} 0 & -ca^3 & ca^2 & 0 & ca^2 & 0 & 0 & -ca^2 \\ -ca^3 & 0 & 0 & ca^3 & 0 & ca^3 & -ca^2 & 0 \\ ca^2 & 0 & 0 & -ca^2 & 0 & -ca^2 & ca & 0 \\ 0 & ca^3 & -ca^2 & 0 & -ca^2 & 0 & 0 & ca^2 \\ ca^2 & 0 & 0 & -ca^2 & 0 & -ca^2 & ca & 0 \\ 0 & ca^3 & -ca^2 & 0 & -ca^2 & 0 & 0 & ca^2 \\ 0 & -ca^2 & ca & 0 & ca & 0 & 0 & -ca \\ -ca^2 & 0 & 0 & ca^2 & 0 & ca^2 & -ca & 0 \end{pmatrix}.$$

The matrices β and $\alpha\beta$ are $\bar{\Lambda}-$Hermitian such that $\alpha\beta = \beta\alpha^*$, the matrices γ and $\gamma\alpha$ are $\Lambda-$Hermitian such that $\gamma\alpha = \alpha^*\gamma$, and $\beta\gamma = \alpha^8$.

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